D-Branes and Geometry

Håkon Enger

Thesis submitted for the degree of
Doctor Scientiarum

Department of Physics
University of Oslo
May 2005

Acknowledgements

This thesis is the result of my PhD studies at the theoretical physics group of the University of Oslo. I would like to thank the group for their support and hospitality, and especially my supervisor Carsten Lütken for taking me as his student even though I had no previous knowledge of the subject, and for his patience and help during these years.

Significant parts of this work has been done at my visits to NORDITA in Copenhagen and the theoretical physics group at King's College London. I would like to thank both of these groups for their hospitality, in particular Paolo Di Vecchia and Andreas Recknagel.

This work has been supported in part by a European Commission Marie Curie Training Site Fellowship under contract no. HPMT-CT-2001-00296.

Contents

I Introduction **1**

1 Introduction **3**
1.1 Motivation . . . 3
1.2 History . . . 4
1.3 String theory . . . 5
1.4 D-branes . . . 11
1.5 Geometry from strings . . . 12
1.6 Geometry of D-branes . . . 19

2 Gauge/gravity for wrapped and fractional branes **21**
2.1 The gauge/gravity correspondence . . . 21
2.2 Wrapped branes . . . 23
2.3 Fractional branes . . . 24
2.4 The Eguchi-Hanson metric . . . 25
2.5 Probing the moduli space of gauge theories . . . 26

3 Stability, topology and geometry **29**
3.1 Sheaves, D-branes and stability . . . 29
3.2 Mumford and Gieseker stability . . . 30
3.3 Π-Stability . . . 31
3.4 Geometric Π-stability . . . 33

4 Gepner models, D-branes and algebraic geometry **35**
4.1 Boundary states in conformal field theory . . . 35
4.2 The Gepner construction . . . 37
4.3 Connection to geometry . . . 39
4.4 Modules and resolutions . . . 40
4.5 Permutation branes . . . 41

A Mathematical Concepts **45**
A.1 Complex and algebraic geometry . . . 45
A.2 Bundles and sheaves . . . 47

Bibliography **49**

II Papers **55**

List of papers

P. Di Vecchia, H. Enger, E. Imeroni and E. Lozano–Tellechea, *Gauge theories from wrapped and fractional branes*, Nucl. Phys. **B 631** (2002) 95-127. [hep-th/0112126]

We compare two applications of the gauge/gravity correspondence to a non conformal gauge theory, based respectively on the study of D-branes wrapped on supersymmetric cycles and of fractional D-branes on orbifolds. We study two brane systems whose geometry is dual to $\mathcal{N} = 4$, $D = 2 + 1$ super Yang–Mills theory, the first one describing D4-branes wrapped on a two-sphere inside a Calabi–Yau two-fold and the second one corresponding to a system of fractional D2/D6-branes on the orbifold $\mathbb{R}^4/\mathbb{Z}_2$. By probing both geometries we recover the exact perturbative running coupling constant and metric on the moduli space of the gauge theory. We also find a general expression for the running coupling constant of the gauge theory in terms of the "stringy volume" of the two-cycle which is involved in both types of brane systems.

H. Enger and C. A. Lütken, *Non-linear Yang-Mills instantons from strings are π-stable D-branes*, Nucl. Phys. **B 695** (2004) 73-83. [hep-th/0312254]

We show that B-type Π-stable D-branes do not in general reduce to the (Gieseker-) stable holomorphic vector bundles used in mathematics to construct moduli spaces. We show that solutions of the almost Hermitian Yang–Mills equations for the non-linear deformations of Yang–Mills instantons that appear in the low-energy geometric limit of strings exist iff they are π-stable, a geometric large volume version of Π-stability. This shows that π-stability is the correct physical stability concept. We speculate that this string-canonical choice of stable objects, which is encoded in and derived from the central charge of the string-*algebra*, should find applications to algebraic geometry where there is no canonical choice of stable *geometrical* objects.

H. Enger, A. Recknagel, and D. Roggenkamp, *Permutation branes and linear matrix factorisations*, In preparation.

We investigate the correspondence between matrix factorisations in Landau-Ginzburg models and in algebraic geometry. We conjecture that there is a correspondence between so-called linear matrix factorisations and permutation branes and present computer-aided case-by-case calculations in evidence for this correspondence.

Part I

Introduction

Chapter 1

Introduction

1.1 Motivation

Geometry is naturally related to string theory since the low energy limit of strings contains Einstein's theory of general relativity (GR), which is a theory for describing the geometry of space and time. One interesting thing about this relation is that GR, and indeed the geometry of space-time itself, arises as a *derived concept* of string theory. String theory can say more about geometry than GR alone, possibly as a consequence of this.

There is hope that string theory can also teach us more about other, more abstract, mathematical objects. A relatively recent discovery of string theory, made in the 1990s, are the *D-branes*. These describe in the low-energy limit a *Yang-Mills (YM) theory* on the part of space-time they cover. A YM theory is a geometrical description of forces such as those appearing in the Standard Model of particle physics. The full mathematical description of a YM theory is a *vector bundle*, or more generally a *sheaf*, objects that mathematicians are working to study and classify. As developments in string theory has given valuable input in the field of algebraic geometry, and *vica versa*, a better understanding of the connection between D-branes and geometry would be helpful in both physics and mathematics.

This thesis seeks to explore the relationship between geometry and D-branes in string theory, in three papers treating some problems where the connection between these worlds is visible. The relationship takes different forms. The first paper concerns a relation found through string theory between the geometry of space-time in a gravitational theory and YM theory, the *gauge/gravity correspondence*. The second paper relates a mathematical stability condition on sheaves and vector bundles to a physical condition of stability of D-branes. The third paper explores the relation of the world-sheet description of string theory, conformal field theory, and the algebraic geometry of sheaves.

The thesis consists of two parts, this introductory text and the three papers. The first part will provide a short introduction to string theory and background for the papers that follow. The general introduction in this first chapter is based mainly on the textbooks [1, 2] and the lecture notes [3, 4]. The next chapters will serve as an introduction to each of the papers in the second part of the thesis. In the appendix, some mathematical concepts, from geometry and algebraic geometry, have been collected for reference.

1.2 History

Modern physics is the name used for the areas of physics that first appeared at the beginning of the 20th century. The two areas of physics covered by this term are Einstein's theory of general relativity (GR), and quantum field theory (QFT). Both have been phenomenally successful in their explanation and prediction of observed phenomena, and many of the tools used daily life in the western world, such as GPS navigation equipment and mobile phones, use technology that would not have been possible without understanding these theories. In fact, with the advantage of hindsight, the prevailing belief of the 19th century that the fundamental physical laws were well understood, and the future of physics was in precision experiments to determine the fundamental natural constants, seems almost absurdly naïve.

The theories of modern physics have also been developed to the point where they lend themselves well to precision experiments. Famous achievements of these theories include the calculation of the orbit of Mercury to more than 8 decimal places and the determination of the fine structure constant α to more than 12 decimal places. Even the age of the cosmos is believed to be known within a few percent. All this could easily lead one to believe that we know what there is to know about fundamental laws of nature.

But there are problems lurking within the dark unexplored corners of modern physics. QFT has intrinsic problems with divergences that appear in the perturbation theory used to make calculations. These divergences are possible to work around as long as the theory is renormalisable, but they still make it difficult to rigorously define most quantum field theories in a mathematical language. Also in GR there are divergences, appearing as the black hole solutions of the field equations for a point mass.

A fundamental problem with the view that the laws of nature are described fundamentally by GR and QFT is that GR as formulated is not a renormalisable theory. This means that a QFT for gravitation must have a different fundamental form than the theory written down by Einstein. Attempts to formulate a theory of gravity compatible with quantum physics have generally not produced satisfactory results, with the exception of one theory which appeared surprisingly from an attempt to solve a different problem.

Before the modern QFT description of the strong force, Quantum Chromodynamics (QCD), was accepted, one proposed theory was that the strong force was mediated by *strings* with a constant tension. This theory had several problems that made it inferior to the theory of QCD, which eventually prevailed. However, in 1974 Scherk and Schwarz [5] showed that string theory had the unexpected power of giving a quantum theory of gravitation. There were still problems, in particular the fact that the appearance of tachyons in the string theory spectrum indicates that the (bosonic) string theory is unstable, but these were finally solved in 1984 when Green and Schwarz [6, 7] proved that the *superstring* theory introduced seven years earlier by Gliozzi, Scherk and Olive [8] consistently removed the tachyons from the spectrum. Thus a combination of QFT and GR without the intrinsic problems encountered before was finally available.

So far, string theory has not been developed to the point where it is completely trustworthy as a fundamental description of the laws of nature. Instead, the theory has again surprisingly proved itself to be useful in different areas from where it was introduced. A mathematical *duality* known as the gauge/gravity correspondence between the open string sector which reduces at low energy to Yang-Mills theory, potentially able to describe the standard model of particle physics, and the closed string sector which reduces to GR, gives promises towards solving the intrinsic problems with divergences in QFT by doing calculations in GR. In the field of mathematics, input from string theory has resulted in a new way of looking at old problems in (algebraic) geometry through the mirror symmetry hypothesis.

1.3 String theory

1.3.1 Introduction

The basic premise of string theory (ST) is the idea of extended fundamental objects. In traditional physics, both classical and quantum, the fundamental building blocks have been assumed to be point particles—zero dimensional objects. ST introduces the concept of the fundamental one dimensional string. In a world governed by the laws of classical physics, the sub-microscopic structure of the building blocks would be of little importance on larger length scales. Perhaps surprisingly, it turns out that the laws of quantum physics constrain heavily the allowed properties of strings and even the space they propagate in.

Another basic assumption usually made in ST is the existence of supersymmetry. This condition turns out to be sufficient in order to state a consistent quantum theory of strings, but it is not known whether it is a necessary condition. So far, no fully consistent theory of non-supersymmetric strings has been formulated.

1.3.2 The string action

In order to formulate a theory for strings, an action must be found from which their equations of motion are derived, and from which a quantum partition function is found. The principle used to formulate such an action is that of extending the action of a point particle in a minimal way. This leads to the Nambu-Goto action: [9, 10]

$$S_{\mathrm{NG}} = -\frac{1}{2\pi\alpha'}\int d\tau d\sigma \sqrt{-\det\gamma}, \quad \gamma_{ab} = \partial_a X^\mu \partial_b X_\mu. \tag{1.1}$$

The numerical value of this action is nothing but the area in space-time of the surface, called the worldsheet, covered by the string as it moves through time. It is a natural analogue of the action for a point particle, $\int ds$, which is the length of the worldline of the particle in space-time.

The Nambu-Goto action is non-trivial to quantise. Quantising it would be easier if the coordinates of the string appeared only as second order in the action. An action with this property may be found, and it is the Polyakov action, [11, 12]

$$S_{\mathrm{P}} = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma \sqrt{-\det g}\, g^{ab}\partial_a X^\mu \partial_b X_\mu. \tag{1.2}$$

The equation of motion for the tensorial metric-like field $h_{\mu\nu}$ from this action is

$$g_{ab}\sqrt{-\det g} = \gamma_{ab}\sqrt{\det\gamma}, \tag{1.3}$$

which determines the field of to a scalar only. The operation of multiplying the metric with a scalar is known as a Weyl rescaling.

$$\text{Weyl:} \quad g_{ab}(x) \mapsto \Omega(x) g_{ab}(x) \tag{1.4}$$

Since this metric is an auxiliary field which gives the Nambu-Goto action when integrated out, the physics can not depend on the value of this scalar. As expected, the Polyakov action is Weyl invariant. It will turn out that Weyl symmetry is in general anomalous in the quantum theory, and this is the main reason for the constraints imposed on the strings and on space-time.

The action (1.2) describes strings moving in a D-dimensional space-time. Nevertheless, it looks like a field theory for the D fields $X^\mu(\tau,\sigma)$ living on the $1+1$-dimensional worldsheet described by coordinates σ and τ. (This is not unique for strings—in the same manner one might describe a point particle with a $0+1$-dimensional field theory.) The field theory on this two-dimensional space-time described by this action is known as a non-linear sigma model. Such models may also be interesting to study in their own rights. The study of string theory thus interacts with the field of two-dimensional physics.

1.3.3 Conformal field theory

The Weyl symmetry of the Polyakov action (1.2) together with its invariance under general coordinate transformations makes it describe a 2-dimensional *conformal field theory* (CFT). A conformal transformation is an automorphism of a manifold which preserves the metric up to a scalar, i.e. induces a Weyl rescaling. An equivalent definition of a conformal transformation is an automorphism that preserves angles. A *local* conformal transformation is a transformation satisfying eq. (1.4) locally. For an infinitesimal transformation $x'^{\mu} = x^{\mu} + \epsilon^{\mu}$, and an initial flat metric $\eta_{\mu\nu}$, the constraint in two dimensions with coordinates x and y becomes,

$$\frac{\partial x'}{\partial x} = \frac{\partial y'}{\partial y} \qquad \frac{\partial x'}{\partial y} = -\frac{\partial y'}{\partial x} \tag{1.5}$$

This is exactly the Cauchy-Riemann conditions for the new coordinate $f(z) = x' + iy'$ to be an analytic function of the old coordinate $z = x + iy$, so in two dimensions a conformal transformation is the same as a holomorphic transformation. This means that using complex coordinates is natural in describing a 2-dimensional CFT.

The following special class of fields will turn out to be important in the study of a CFT: A *primary field* is a field ϕ which transforms according to

$$\phi(z, \bar{z}) \mapsto \left(\frac{\partial f}{\partial z}\right)^{h} \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \phi(f(z), \bar{f}(\bar{z})) \tag{1.6}$$

under the conformal transformation $z \mapsto f(z)$. Such a field is said to have *conformal weight* $(h, \bar{h})$. Since the transformation factorises into an analytic and an anti-analytic part, it is practical to view the transformations $z \mapsto f(z)$ and $\bar{z} \mapsto \bar{f}(\bar{z})$ as independent of each other.

Conformal invariance of a *quantum* field theory requires that the correlation functions are conformally invariant. Since coordinate transformations are generated by the energy-momentum tensor $T^{\mu\nu}$, this becomes the requirement that for any primary field ϕ,[1]

$$T(z)\phi(w) = \frac{h}{(z-w)^2}\phi(w, \bar{w}) + \frac{1}{z-w}\partial_w \phi(w, \bar{w}) + \ldots, \tag{1.7}$$

where the dots represent terms regular as $z \to w$, and the equation is an example of an *operator product expansion*, which is to be understood as an identity only inside correlation functions. The energy momentum tensor itself turns out not to be a primary field in general, instead having an operator product with itself on the form

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w) + \ldots. \tag{1.8}$$

The constant c is the *central charge* of the CFT.

[1] As is common in conformal field theory, the tensor component $T^{zz}(z)$ is called just $T(z)$ and the component $T^{\bar{z}\bar{z}}(\bar{z})$ will be called $\bar{T}(\bar{z})$.

1.3.4 Supersymmetry

There exists a natural supersymmetrisation of the Polyakov action (1.2), by adding terms for the fermions to the action. [3]

$$S = \frac{1}{4\pi}\int d^2\sigma \left(\frac{1}{\alpha'}\partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial}\psi_\mu + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu\right), \tag{1.9}$$

where $\psi, \tilde{\psi}$ are fermion fields. There may be several reasons to introduce supersymmetry. First, we need to include fermions somehow in order to make a realistic theory of nature. They may be added in a non-supersymmetric way, of course, but extending the conformal symmetry of the non-linear sigma model to a superconformal one may be argued to be a "minimal" enlargement of the theory.

Perhaps more importantly, the tachyon of the bosonic string is a serious problem for its usefulness. It turns out that these exists a consistent introduction of supersymmetry eliminates the tachyon, by projecting out states with odd *fermion number*. This is called the Gliozzi-Scherk-Olive (GSO) projection and ensures *spacetime* supersymmetry.

1.3.5 Closed strings

The equation of motion following from (1.2) is the Laplacian (wave equation) on the space with metric $g_{\mu\nu}$. If the topology of the worldsheet is that of a torus, or that of flat affine space, we may use the symmetries of the action to transform the metric to the form $g_{\mu\nu} = \eta_{\mu\nu}$. (In other cases, we may do this only locally.) The equations of motion then become

$$(\partial_\sigma^2 - \partial_\tau^2)X^\mu = 0, \tag{1.10}$$

with solutions $X^\mu(\tau,\sigma) = X_R^\mu(\tau-\sigma) + X_L^\mu(\tau+\sigma)$,

$$X_R^\mu(\sigma^-) = \frac{1}{2}x^\mu + \sqrt{\frac{\alpha'}{2}}\alpha_0^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-2in\sigma^-}, \tag{1.11}$$

$$X_L^\mu(\sigma^+) = \frac{1}{2}x^\mu + \sqrt{\frac{\alpha'}{2}}\tilde{\alpha}_0^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-2in\sigma^+}, \tag{1.12}$$

We identify the coefficient of τ as being proportional to the (centre of mass) momentum, $p^\mu = \sqrt{\frac{1}{2\alpha'}}(\alpha_0^\mu + \tilde{\alpha}_0^\mu)$. We will here impose periodic boundary conditions in σ, appropriate for describing a *closed* string. The condition in flat space becomes $X^\mu(\sigma,\tau) = X^\mu(\sigma+2\pi,\tau)$, giving $\alpha_0^\mu = \tilde{\alpha}_0^\mu$. If we assume that (at least) one dimension, say X^{25}, is periodic (we will say that it is *compactified* on a circle) with period $2\pi R$, we can have

$$\alpha_0^\mu - \tilde{\alpha}_0^\mu = \sqrt{\frac{2}{\alpha'}}wR, \tag{1.13}$$

where w is an integer we identify as the *winding number* of the string around the periodic dimension.

For the superstring action, eq. (1.9), we need boundary conditions for the fermions as well. In this case, the boundary conditions needs only to be periodic up to a sign in order to be consistent with the equations of motion. Periodic conditions are known as the Ramond (R) sector, while anti-periodic conditions are known as the Neveu-Schwarz (NS) sector,

$$\text{R:} \quad \psi^\mu(\sigma,\tau) = \psi^\mu(\sigma+2\pi,\tau), \tag{1.14}$$

$$\text{NS:} \quad \psi^\mu(\sigma,\tau) = -\psi^\mu(\sigma+2\pi,\tau). \tag{1.15}$$

The same choice may be made for the right-moving $\tilde{\psi}$ field, leading to four possibilities all in all, R–R, R–NS, NS–R, and NS–NS. The GSO projection removing the tachyon from the spectrum may be made in two different ways in the R sector, and this gives two different consisting closed superstring theories, called *type IIA* and *type IIB*.

1.3.6 Open strings

In the previous section we applied periodic boundary conditions to the equations of motion for a string, and got a theory for a closed string. We found that in this case, the left and right moving oscillations were independent of each other. There are two other possibilities for boundary conditions that we will study, the Neumann and Dirichlet conditions. Both these result in open strings. In the Neumann case, the ends of the string are free to move, and in the Dirichlet case the endpoints are fixed on a subspace (submanifold) of space-time. We will see that in the open case, the left and right moving oscillations are not independent.

For open strings, it is customary to let σ run from 0 to π. The Neumann boundary conditions read $X'^\mu(\tau,0) = X'^\mu(\tau,\pi) = 0$. In this case the general solution to the equation of motion (1.10) is

$$X^\mu(\tau,\sigma) = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos(n\sigma). \tag{1.16}$$

The other possibility we should consider is Dirichlet boundary conditions, $\dot{X}^\mu(\tau,0) = \dot{X}^\mu(\tau,\pi) = 0$. This means that the endpoints of the string is fixed. The solution becomes

$$X^\mu(\tau,\sigma) = x_0^\mu + \frac{1}{\pi}x_\pi^\mu\sigma + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\sin(n\sigma). \tag{1.17}$$

The string is now stretched between the points x_0^μ and x_π^μ. It is also possible to combine Neumann boundary conditions in some directions and Dirichlet conditions in other directions. Immediately, it seems that the Dirichlet condition doesn't make sense (e.g., it's not Lorentz invariant) unless there is some object to which the string is connected. This object is the D-brane, which will be the main topic of this thesis.

For the superstring action, there are again two possible choices consistent with the equations of motion, Ramond and Neveu-Schwarz,

$$\text{R:} \quad \psi^\mu(0,\tau) = \tilde{\psi}^\mu(0,\tau) \qquad \psi^\mu(\pi,\tau) = \tilde{\psi}^\mu(\pi,\tau) \tag{1.18}$$

$$\text{NS:} \quad \psi^\mu(0,\tau) = -\tilde{\psi}^\mu(0,\tau) \qquad \psi^\mu(\pi,\tau) = \tilde{\psi}^\mu(\pi,\tau) \tag{1.19}$$

(1.20)

In this case the two different consistent GSO projections in the R sector turn out to give equivalent results, so there is only one consistent open superstring theory, the *type I* theory.

1.3.7 Faddeev-Popov ghosts

We would like to gauge out the Weyl symmetry of the QFT partition function, by removing the gauge degrees of freedom from the functional integral $Z = \int \mathcal{D}X e^{-S}$. This is commonly done with the *Faddeev-Popov procedure.* The gauge is fixed by doing a gauge transformation to a chosen metric, which we may simply choose as the Minkowski metric $\eta_{\mu\nu}$. However, since the functional measure $\mathcal{D}X$ is not gauge invariant, this transformation introduces a Jacobian into the integral which is the Faddeev-Popov determinant. This determinant may be represented as the partition function for so-called *ghost* fields b, c resulting in a total partition function

$$Z = \int \mathcal{D}X\mathcal{D}b\mathcal{D}c e^{-S^\eta - S_g}, \tag{1.21}$$

where S^η is the gauge fixed action and S_g is the action for the ghost fields,

$$S_g = \frac{1}{2\pi} d^2z(b\bar{\partial}c + \bar{b}\partial\bar{c}), \tag{1.22}$$

where b and c are fields with conformal weight $(2,0)$ and $(-1,0)$, respectively. The central charge for the ghost theory may be calculated to $c_g = -26$.

The fermions in the supersymmetric string theory also introduce ghosts into the partition function, bringing the total ghost central charge to $c_g = -15$.

1.3.8 T-duality

Going back to the case of one dimension compactified on a circle, we know from basic quantum mechanics that in this case the momentum in the compact dimension will be quantised as well,

$$p^\mu = \frac{1}{\sqrt{2\alpha'}}(\alpha_0^\mu + \tilde{\alpha}_0^\mu) = \frac{n}{R}, \tag{1.23}$$

where n is an integer.

The mass spectrum may be computed by

$$m^2 = -p^\mu p_\mu = \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2), \tag{1.24}$$

where $N + \tilde{N}$ is the total number of excitations of the oscillators $\alpha^\mu_m, \tilde{\alpha}^\mu_m$. This equation is invariant under the exchange

$$n \leftrightarrow w, \quad R \leftrightarrow \frac{\alpha'}{R} =: R'. \tag{1.25}$$

This is the sign of *T-duality*. In fact, the whole partition function of the theory is invariant under this action: The partition function is [3]

$$Z(q, R) = (\eta\bar{\eta})^{-1} \sum_{n,w} q^{(\alpha_0^{25})^2/2} q^{(\tilde{\alpha}_0^{25})^2/2}, \tag{1.26}$$

and since the only action of T-duality is to change the sign of $\tilde{\alpha}_0^{25}$, this function in clearly invariant.

The T-dual theory may be interpreted as a similar theory with a coordinate $X'^{25}(\sigma, \tau) := X_R^{25}(\sigma + \tau) - X_L^{25}(\sigma - \tau)$, compactified on a circle with the dual radius R'.

1.4 D-branes

Historically, the possibility of the Dirichlet boundary conditions was initially overlooked since it breaks coordinate invariance to fix a coordinate to a specific value. However, if the subspace in which the coordinates are free is interpreted as a physical and dynamical object, a D(irichlet)-brane, the boundary condition is compatible with relativity. In this case, the open strings are connected to D-branes at their end points.

The Dirichlet boundary condition must also be included as a possibility if we are to make sense of T-duality for open strings. For an open string, the action of T-duality exchanges Neumann and Dirichlet boundary conditions in the direction corresponding to the T-dualised coordinate. Since the Dirichlet boundary condition only makes sense if there is an object to which the string is connected, this means that the T-duality must also introduce such an object, a D-brane, into the theory.

There exist p-dimensional (mem)brane solutions in supergravity theories, i.e. solutions of the supergravity equations corresponding to p-dimensional sources. These solutions are also charged under the R–R fields. In 1995, Polchinski [13] realized that the extremal versions of these branes were the low-energy limit of the Dirichlet branes from string theory. The extremal metric, dilaton, and R–R form field are [3]

$$ds^2 = H_p^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + H_p^{1/2}dx^i dx^i, \tag{1.27}$$

$$e^{2\Phi} = g_s^2 H_p^{\frac{3-p}{2}} \tag{1.28}$$

$$C_{(p+1)} = (H_p^{-1} - 1)g_s^{-1}dx^0 \wedge \cdots \wedge dx^p, \tag{1.29}$$

where the coordinates labelled with Greek indices are along the world-volume of the brane and those with Latin indices are perpendicular to it. H_p is an harmonic function $H_p = 1 + \alpha_p g_s N/r^{6-p}$ where α_p is a constant.

D-branes are *BPS states*. This implies that a D-brane preserves half of the supersymmetry of the theory (A collection of D-branes may break more or all of the SUSY.)

1.5 Geometry from strings

1.5.1 Number of dimensions

As mentioned earlier, the symmetries and consistency demands of ST gives restrictions on the target space geometry and topology. The most basic restriction is that of the number of dimensions.

The common theme of the various restrictions we will find is that of Weyl invariance of the Polyakov action (1.2). As explained above, this symmetry is required for the Polyakov action to make sense physically. Furthermore, scale invariance is a nice property for a proposed fundamental theory.

To show the restriction on dimension[2], it is enough to consider a flat target space metric $G_{\mu\nu} = \eta_{\mu\nu}$ and a perturbation of a flat worldsheet metric, $g_{ab} = \eta_{ab} + \delta g_{ab}$. By definition of the energy-momentum tensor T_{ab}, the response of the partition function to such a perturbation is

$$\delta Z = -\frac{1}{4\pi}\int d^2\sigma \delta g_{ab}(\sigma,\tau)\langle T^{ab}(\sigma,\tau)\rangle \tag{1.30}$$

and for Weyl symmetry, this response should vanish under a Weyl transformation, $\delta g_{ab} = 2\omega\eta_{ab}$. We must thus calculate the expectation value of the energy-momentum tensor, and demand that its trace vanishes. Perturbatively, this looks easy since it again involves simply inserting another T_{ab}:

$$\delta\langle T^{ab}(\sigma,\tau)\rangle = -\frac{1}{4\pi}\int d^2\sigma' \delta g_{cd}(\sigma',\tau')\langle T^{ab}(\sigma,\tau)T^{cd}(\sigma',\tau')\rangle. \tag{1.31}$$

Going to complex coordinates, we know from conformal field theory that $\langle T_{zz}(z)T_{zz}(z')\rangle = \frac{c}{2}(z-z')^4$. In order for this result to make sense in the above integral, it is necessary to insert a cut-off a at small distances $|z-z'|$:

$$\delta\langle T_{zz}(z)\rangle = -\frac{c}{8\pi}\int d^2z' \frac{\delta g^{zz}}{(z-z')^4}\theta(|z-z'|^2 - a^2), \tag{1.32}$$

[2] The following is based on Cardy [14] and Polchinski [15, ch. 3]

where θ is a step function. Note that this cut-off breaks the conformal invariance. Taking the antiholomorphic derivative of the above,

$$\partial_{\bar{z}}\delta\langle T_{zz}(z)\rangle = -\frac{c}{8\pi}\int d^2z'\frac{\delta g^{zz}(z')}{(z-z')^3}\delta(|z-z'|^2-a^2) = -\frac{c}{48}\partial_z^3\delta g^{zz}(z), \quad (1.33)$$

where we have made a Taylor expansion of $g^{zz}(z')$ around z and noted that the other terms of the expansion does not survive the angular integration.

The result (1.33) is not diff-invariant, and can therefore not be the whole solution. Ordinarily in the conformal theory, the contributions from other terms such as $\langle T_{zz}(z)T_{z\bar{z}}(z')\rangle$ would be zero, however because the cut-off introduced breaks conformal invariance, this is not the case here. These contributions restore diff-invariance, and we may deduce their form by adding the necessary terms for invariance to (1.33). We can also use the continuity equation for the energy-momentum tensor,

$$\partial_{\bar{z}}T_{zz} = -\partial_z T_{\bar{z}z} \quad (1.34)$$

to find the expression for $T_{\bar{z}z}$ (needed for the trace), resulting in

$$\delta\langle T_{\bar{z}z}(z)\rangle = -\frac{c}{48}\left(\partial_z^2\delta g^{zz} - 2\partial_{\bar{z}}\partial_z\delta g^{\bar{z}z} + \partial_{\bar{z}}^2\delta g^{\bar{z}\bar{z}}\right) = -\frac{c}{48}R, \quad (1.35)$$

where R is the Ricci scalar (correct to first order in δg^{ab}) of the worldsheet.

To make the theory invariant (anomaly free) under the Weyl symmetry, one must therefore have $c = 0$. Each coordinate field X^μ contributes $c = 1$ in bosonic string theory and $c = 3/2$ in superstring theory. The contribution from the Fadeev-Popov ghosts, $c = -26$ and $c = -15$, respectively, means that this implies $D = 26$ for the bosonic string and $D = 10$ for the superstring.

1.5.2 Effective gravity

In the low-energy limit, the stringy structure will not be visible directly. Instead, the strings will look like point particles, with spin and mass depending on their internal structure. Only the massless excitations will enter in the effective low-energy action.

The massless spin-2 particles coming from the closed strings are natural candidates for gravitons. To see whether it actually resembles a graviton in the low energy limit, we must find its effective action. In principle, one might do this by integrating out the high energy modes from the action (1.2). However, Weyl invariance restricts the form of the action enough to make this procedure unnecessary.

Integrating out the high-energy part of the Polyakov action, eq. (1.9), will leave an effective action on the form of a *nonlinear sigma model*

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-\det g} \Big[\Big(g^{ab} G_{\mu\nu}(X) + \epsilon^{ab} B_{\mu\nu}(X) \Big) \partial_a X^\mu \partial_b X^\nu + \alpha' R \Phi(X) \Big], \quad (1.36)$$

where $G_{\mu\nu}$ is a symmetric field which we recognise as a target space metric, $B_{\mu\nu}$ is an antisymmetric field and Φ is a target space scalar. Demanding Weyl invariance of this action turns out to give the equations, to first order in α',

$$\alpha' \mathbf{R}_{\mu\nu} + 2\alpha' \nabla_\mu \nabla_\nu \Phi - \frac{\alpha'}{4} H_{\mu\lambda\omega} H_\nu{}^{\lambda\omega} = 0, \quad (1.37)$$

$$-\frac{\alpha'}{2} \nabla^\omega H_{\omega\mu\nu} + \alpha' \nabla^\omega \Phi H_{\omega\mu\nu} = 0, \quad (1.38)$$

$$\frac{D-26}{6} - \frac{\alpha'}{2} \nabla^2 \Phi + \alpha' \nabla_\omega \Phi \nabla^\omega \Phi - \frac{\alpha'}{24} H_{\mu\nu\lambda} H^{\mu\nu\lambda} = 0, \quad (1.39)$$

where $\mathbf{R}_{\mu\nu}$ is the target space Ricci tensor, $H = dB$, and ∇_μ is the covariant derivative on the target space. We recognise the first of these as Einstein's equations with source terms from $B_{\mu\nu}$ and Φ, but note that it receives corrections in higher order in α'. The effective space-time action corresponding to these equations of motion is

$$S = \frac{1}{2\kappa^2} \int d^{26}x \sqrt{-\det G} e^{-2\phi} \left[\mathbf{R} - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + 4\partial_\mu \Phi \partial^\mu \Phi + \mathcal{O}(\alpha') \right]. \quad (1.40)$$

Extending the nonlinear sigma model above with fermions to make it supersymmetric will introduce more space-time fields. The fields in the NS–NS sector correspond to the fields in the bosonic case above. In addition, we get fermionic fields (coming from the R–NS and NS–R) sector, and additional bosonic fields from the R–R sector. The latter fields organise into antisymmetric p-tensors $C_{(p)}$, where p is odd in type IIA theory and even in type IIB theory. The result is the type IIA and IIB *supergravity* theories, with bosonic actions

$$S_{\text{IIA}} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-\det G} \left\{ e^{-2\Phi} \left[R + 4(\nabla\Phi)^2 - \frac{1}{2} H_{(3)}^2 \right] - \frac{1}{4} G_{(2)}^2 - \frac{1}{48} G_{(4)}^2 \right\} - \frac{1}{4\kappa^2} \int B_{(2)} dC_{(2)} dC_{(3)} \quad (1.41)$$

$$S_{\rm IIB} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-\det G}\Bigg\{e^{-2\Phi}\left[R + 4(\nabla\Phi)^2 - \frac{1}{2}H_{(3)}^2\right]$$
$$- \frac{1}{12}(G_{(3)} + C_{(0)}H_{(3)})^2 - \frac{1}{2}(dC_{(0)})^2 - \frac{1}{480}(G_{(5)})^2\Bigg\}$$
$$+ \frac{1}{4\kappa^2}\int\left(C_{(4)} + \frac{1}{2}B_{(2)}C_{(2)}\right)G_{(3)}H_{(3)}, \quad (1.42)$$

where the field strengths are

$$\begin{array}{ll} H_{(3)} = dB_{(2)}, & \\ G_{(2)} = dC_{(1)}, & G_{(4)} = dC_{(3)} + H_{(3)} \wedge C_{(1)}, \\ G_{(3)} = dC_{(3)}, & G_{(5)} = dC_{(4)} + H_{(3)} \wedge C_{(2)}. \end{array} \quad (1.43)$$

1.5.3 Gauge theory

Gauge theory from open strings

Since the open string theory contains strings transforming as vectors in space-time, this theory might give a model of a gauge theory, or more specifically of a Yang-Mills theory at low energy. To find the effective action, one must work out the amplitudes for string scattering, and this leads to an Abelian (U(1)) gauge theory at low energy.

To allow for other gauge groups, more degrees of freedom must be introduced by hand into the string action. These so-called Chan-Paton factors [16] may be interpreted as discrete degrees of freedom associated with the endpoints of the string. The low-energy space-time action receives a term $-\frac{1}{4g_{\rm YM}^2}\,{\rm Tr}(F^2)$, where $F^{\mu\nu}$ is the matrix-valued Yang-Mills field strength.

The gauge group of this theory depends on the number of degrees of freedom in the Chan-Paton factors, and on whether the strings are oriented or not. For oriented strings, the gauge group will be U(N), while in the unoriented case, the group will be SO(N) or Sp($N/2$). However, the oriented open string theory may be shown to be to be inconsistent.

It turns out that the gauge group is restricted even more. In the type I superstring theory, the combined R-R part of the partition function for the cylinder, the Möbius strip and the Klein bottle turns out to be [2, ch. 10]

$$Z = -i(N \pm 32)^2 C \int_0^\infty ds(1 + \mathcal{O}(e^{-2s})), \quad (1.44)$$

with the upper sign for Sp($N/2$) and the lower one for SO(N). This diverges[3] badly unless the gauge group is exactly SO(32).

[3]The NS-NS part of the partition function cancels the R-R part because of supersymmetry, but when an amplitude is calculated with vertex operators inserted, this divergence will actually matter.

Gauge theory from D-branes

To find the low-energy action of a D-brane, one may again look at the condition for Weyl invariance, for a sigma model on a worldsheet with boundaries at $\sigma = 0, \pi$. This procedure will give space-time equations of motion, leading to the following action:

$$S_{\text{DBI}} = -T_p \int d^{p+1}\xi e^{-\Phi} \det \sqrt{G_{\mu\nu} + B_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}}, \tag{1.45}$$

where ξ^μ are coordinates on the D-brane world volume, and T_p is the tension of the D-brane. This action is known as the Dirac-Born-Infeld action. By expanding it in α', we see that it is a deformation of the Yang-Mills action,

$$S = -T_p \int d^{p+1}\xi e^{-\Phi} \operatorname{Tr} F_{\mu\nu}F^{\mu\nu} + \mathcal{O}(\alpha'), \tag{1.46}$$

so the low-energy limit is indeed again a Yang-Mills theory, this time restricted to the $p+1$ dimensions spanned by the D-brane. This fact has lead to the so-called "brane world" models where our observable universe is conjectured to be a D3-brane.

From Dp-branes in type I superstring theory, we get the DBI action, eq. (1.45), plus a part from the interaction with the R–R field $C_{(p)}$ called a Wess-Zumino part,

$$S_{\text{WZ}} = \mu_p \int C_{(p)} \wedge e^{2\pi\alpha' F + B}, \tag{1.47}$$

Including instanton effects, it can be shown by an anomaly inflow argument [17, 18] that the corrected version is

$$S = S_{\text{DBI}} + \mu_p \int C_{(p)} \wedge e^{2\pi\alpha' F + B} \sqrt{\hat{\mathcal{A}}(4\pi\alpha' R)}, \tag{1.48}$$

where $\hat{\mathcal{A}}(R)$ is a polynomial known as the *A-roof genus*.

Gauge theory from compactification

The Kaluza-Klein method describes how gauge bosons appear when some dimensions are wrapped on a small compact space. Consider the effect of compactifying one dimension on a circle on the gravitational theory. A general metric may be written on the form

$$ds^2 = G^{26}_{MN} dx^M dx^N = G^{25}_{\mu\nu} dx^\mu dx^\nu + e^{2\sigma}(dx^{26} + A_\mu dx^\mu)^2. \tag{1.49}$$

The Ricci scalar for this metric becomes $\mathbf{R}_{26} = \mathbf{R}_{25} - 2e^{-\sigma}\nabla^2 e^\sigma - \frac{1}{4}e^{2\sigma} F_{\mu\nu}F^{\mu\nu}$, and so the 25-dimensional theory becomes a theory of gravitation plus a gauge field A_μ and a dilaton constructed from σ.

By compactifying several dimensions, non-Abelian Yang-Mills theories will appear after the compactification. The symmetries of the space on which the original theory is compactified will determine the gauge group of the YM theory.

Enhanced gauge symmetry from singularities

One of the major new developments of string theory in the 1990s was the realization that the five consistent superstring theories were different aspects of the same theory. In particular, there is an explicit duality between Type IIA string theory and the heterotic string theory, which has an enhanced gauge theory at certain points of its moduli space. This enhancement should also exist in the IIA moduli space.

The enhanced gauge theory exist on points where the theory is geometrically singular. The singularity reflects that certain modes become massless, which enable the enhancement of the symmetry. E.g., on A_{n+1}-singularities, the gauge group is $U(n)$.

The process of the enhanced gauge symmetry was explained [19] in the T-dual picture of IIB ST (more precisely, the T-duality should be combined with an S-duality to get the picture described here), where the A_{n-1} case is mapped to n D1-branes approaching each other on the singular point in the moduli space. In this picture, the gauge theory enhancement is the standard Chan-Paton mechanism where the $U(1)^n$ gauge group is enhanced to $U(n)$.

1.5.4 Supersymmetry and geometry

World sheet supersymmetry

Demanding *extended* supersymmetry of the world sheet action has consequences for the allowed target space geometry. For the case where the B-field is closed and the dilaton Φ constant, $\mathcal{N} = 2$ supersymmetry (counting both left- and right-moving sectors, $\mathcal{N} = (2,2)$ SUSY) means [20] that the target space must be a complex Kähler manifold. One supersymmetry transformation may be normalised such that it has the form $\delta_\epsilon X^\mu = \bar{\epsilon}\psi^\mu$. The extended supersymmetry will then be of the form

$$\delta_\epsilon X^\mu = \bar{\epsilon} f^\mu{}_\nu X^\nu, \tag{1.50}$$

and it turns out that the tensor $f^\mu{}_\nu$ must be covariantly constant and satisfy

$$f^\mu{}_\nu f^\nu{}_\lambda = -\delta^\mu{}_\lambda, \qquad G_{\mu\nu} f^\mu{}_\lambda f^\nu{}_\omega = G_{\lambda\omega} \tag{1.51}$$

for this to be consistent. These equations imply that $f^\mu{}_\nu$ defines an almost complex structure on the target space, and $G_{\mu\nu}$ is a Hermitian metric. Since the almost complex structure is covariantly constant, the manifold is Kähler.

Target supersymmetry

It is one thing to require a symmetry of an action, another thing to require the symmetry of a state. If we want the vacuum state to preserve (some of the) supersymmetry, this will restrict the action even more.

We will look at the effective low-energy fields, specifically the fermions, and require that they are invariant under a supersymmetry transformation. The supersymmetry transformation of the gravitino ψ the dilatino χ and the gaugino λ turn out to be

$$\delta\psi_\mu = \nabla_\mu \epsilon \tag{1.52}$$

$$\delta\psi_m = (\partial_m + \frac{1}{4}\Gamma_{mnp}\gamma^{np})\epsilon \tag{1.53}$$

$$\delta\chi = (\gamma^m \partial_m \phi)\epsilon \tag{1.54}$$

$$\delta\lambda = F_{mn}\gamma^{mn}\epsilon, \tag{1.55}$$

where Latin indices are used for the flat space-time directions and Greek indices for the compactified directions. $\epsilon(x)$ is an infinitesimal supersymmetry parameter. The action is invariant under any choice of ϵ, but the first equation above tells us that ϵ must be covariantly constant in the compact space in order to be a symmetry of the vacuum state.

The compact manifold on which we compactify the extra six dimensions must therefore allow a covariantly constant spinor. This is not true for all manifolds. Furthermore, if the manifold in question allows several covariantly constant spinors, our vacuum state will have extended supersymmetry.

To see what this means for the topology of the 6-manifold, consider the holonomy group $G \subseteq SO(6)$. Since there is a covariantly constant spinor ϵ, this group is such that $g\epsilon = \epsilon$ for all $g \in G$. The maximal subgroup of $SO(6)$ that has this property is $SU(3)$. If the holonomy is a proper subgroup of $SU(3)$, there will be several constant spinors and extended supersymmetry. For the type II string, the left- and right-moving sectors will each provide one space-time supersymmetry generator, giving $\mathcal{N} = 2$ in the case where the holonomy group is the full $SU(3)$.

An n-dimensional Kähler manifold with $SU(n)$ holonomy is called a *Calabi-Yau* manifold. The condition on holonomy is equivalent to the condition of a *vanishing first Chern class.* Is was conjectured by Calabi and proven by Yau that such manifolds always admit a unique Ricci-flat metric. This means that the Einstein equations on such a manifold have an unique solution. However, since, as we have seen, there are corrections to these equations from string theory, the correct "stringy" metric is a deformation of this one.

There are a vast amount of (families of) Calabi-Yau manifolds. It is not currently known whether the number of families is finite or not. Many such manifolds, or *varieties*, may be found as solutions of polynomial equations in projective space. One family has been much studied in the literature, the *Quintic* defined by a degree five polynomial in $\mathbb{CP}^4$. Of special interest is the *Fermat quintic*

$$x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = 0, \tag{1.56}$$

where the x_i are homogeneous coordinates on $\mathbb{CP}^4$.

1.6 Geometry of D-branes

1.6.1 Boundary Conditions

The $\mathcal{N} = 2$ supersymmetric sigma model was used by Ooguri, Oz, and Yin[21] to classify the possible boundary conditions consistent with a boundary representing a D-brane in spacetime, i.e. a BPS state preserving half of the supersymmetry. The consistent boundary conditions were grouped into *A-type* boundary conditions

$$J_L = -J_R, \quad G_L^+ = \pm G_R^-, \quad U_L = e^{i\theta} U_R^*, \tag{1.57}$$

and *B-type* boundary conditions

$$J_L = J_R, \quad G_L^+ = \pm G_R^+, \quad U_L = \pm U_R. \tag{1.58}$$

Here, $J_{L,R}$ are the $U(1)$ currents, $G_{L,R}^{\pm}$ are the supersymmetry generators, and $U_{L,R}$ is the spectral flow operator. The factor θ may be absorbed into the definition of the holomorphic n-form on the CY, but we will keep it explicit here. If there are several A-type D-branes present, they will not define a BPS state unless they have the same value for θ. In the language of Fuchs et al.[22], an A-type boundary condition with a specified value for θ may be called an A_θ-type boundary condition.

In general, the boundary conditions may be written

$$\partial X^\mu = R^\mu{}_\nu \bar{\partial} X^\nu, \tag{1.59}$$

for an orthogonal matrix R. Eigenvectors of R with eigenvalue $+1$ and -1 correspond to Neumann and Dirichlet boundary conditions, respectively. The A-type condition (1.57) implies

$$\Omega_{\mu_1 \ldots \mu_n} R^{\mu_1} \cdots R^{\mu_n} = e^{i\theta} \bar{\Omega}_{\nu_1 \ldots \nu_n} \tag{1.60}$$

1.6.2 Special Lagrangian Cycles

The A-type boundary condition was shown by Ooguri et al. to imply that the D-brane wraps a *special Lagrangian* cycle, the same condition found by Becker et al.[23] for A-type D-branes from the low-energy effective supermembrane action. The condition that a cycle D in the CY space X is special Lagrangian is that[24]

$$J|_D = 0, \qquad \operatorname{Im} e^{i\theta} \Omega|_D = 0, \tag{1.61}$$

where J is the Kähler form and Ω is the holomorphic n-form on X.

The cycle D may be expanded in a symplectic basis (A^i, B_i) as

$$D = Q_i A^i + \tilde{Q}^i B_i. \tag{1.62}$$

The integrals of the holomorphic n-form over the basis cycles determines the *periods*

$$z^i = \int_{A^i} \Omega \qquad F_i = \int_{B_i} \Omega. \tag{1.63}$$

1.6.3 B-type D-branes

Ooguri et. al.[21] (see also [23]) found that a D-brane with a "B-type" boundary condition is wrapping a *holomorphic submanifold* of the Calabi-Yau space X. In particular, this means that the submanifold is even dimensional.

In addition, there are conditions on the gauge field living on the brane[4]. The conditions for unbroken supersymmetry in a gauge theory was first studied in the context of the heterotic string, see [25, ch. 15]. In this case, with the assumption that $H = dB = 0$ and a constant dilaton, the conditions on the field strength may be given in complex coordinates as

$$F_{\mu\nu} = F_{\bar{\mu}\bar{\nu}} = 0, , \tag{1.64}$$

$$g^{\mu\bar{\nu}} F_{\mu\bar{\nu}} = 0, \tag{1.65}$$

where $F_{\mu\bar{\nu}} = F^a_{\mu\bar{\nu}} t_a$ is the Yang-Mills field strength and $g^{\mu\bar{\nu}}$ is the (Hermitian) metric on X. Equation (1.64) says that F is a $(1,1)$-form, and this is equivalent to saying that the vector bundle E (defined by the field strength F) is holomorphic.

Equation (1.65) is correct to lowest order in α' only. A more detailed discussion was done by Mariño et al. [26] by studying the conditions for instantons in SYM to be BPS and by Kapustin [27] from the worldsheet point of view. The result for a brane wrapping a 6-dimensional Kähler manifold with $B = 0$ is

$$\omega \wedge \omega \wedge F - \frac{1}{3}(F \wedge F \wedge F) = \cot \theta_E (\omega \wedge F \wedge F - \frac{1}{3} \omega \wedge \omega \wedge \omega), \tag{1.66}$$

where ω is the Kähler form of the manifold and θ is a constant which may be determined by the topological charges of the D-brane.

[4]This is the low-energy view point, of course, the "gauge field" in the high-energy limit should be replaced by something "stringy".

Chapter 2

Gauge/gravity for wrapped and fractional branes

2.1 The gauge/gravity correspondence

2.1.1 D-Branes and open/closed duality

From the point of view of open string theory, D-branes are objects to which the strings' endpoints may be attached. We know from section 1.5.3 that in the low energy limit there is a Yang-Mills theory living on the world-volume of the D-brane.

Now consider the closed string theory. In this picture, D-branes are in the low-energy limit solutions of supergravity corresponding to massive objects with Ramond-Ramond charge. This limit (apparently) does not contain a Yang-Mills theory. However, by considering open/closed duality, we know that these two pictures should describe the same phenomena. This is the basic idea behind the gauge/gravity correspondence.

To make this idea more precise, we should give a more precise definition of the low-energy limit. The limit should be taken such that the sector of the open string theory which interacts with the D-brane is separated from the free closed string sector which exists far from the brane. This introduction is based on the review [28].

2.1.2 The AdS/CFT correspondence

The AdS/CFT correspondence is the gauge/gravity correspondence for the specific case of a D3 brane in flat space. It was discovered in the low energy limit of supergravity [29].

Consider this limit of a string theory containing N overlapping D3 branes. The low energy effective string action can be written schematically as

$$S = S_{\text{bulk}} + S_{\text{brane}} + S_{\text{int}}. \tag{2.1}$$

S_{bulk} is the low energy limit of closed string theory, 10-dimensional (type IIB) supergravity. S_{brane} is the limit of open string theory, the DBI action on the brane world-volume, which to zeroth order in α' is an $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory. S_{int} collects terms in the low energy action with interactions between these two sectors.

The bulk Lagrangian may be expanded in the square root of Newton's constant, $\kappa \sim g_s \alpha'^2$, and is to zeroth order in this parameter a free theory. The interaction part of the action is proportional to κ to leading order. Thus, if we keep the (low) energy fixed and send $\alpha' \to 0$, the system decouples into a free gravitational theory and the four dimensional SYM theory.

From the supergravity point of view, a D-brane is described by the solution given in eq. (1.27) with the metric

$$ds^2 = H^{-1/2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + H^{1/2}(dr^2 + r^2 d\Omega_5^2). \tag{2.2}$$

The energy of objects close to the horizon is redshifted from the point of view of an observer far away. In the low energy limit, excitations close to the horizon will not have the energy available to interact with excitations far from the brane. These two regions decouple in the limit, leaving free bulk gravity (as above), and the near horizon limit of the geometry (2.2) which may be approximated by $H \sim R^4/r^4$, leaving

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{R^2}{r^2} dr^2 + R^2 d\Omega_5^2, \tag{2.3}$$

which is the metric for the space $\text{AdS}_5 \times S^5$. The proper near horizon limit to take is $r \to 0$ with r/α' fixed, such that the energy of objects in the near-horizon region is fixed (with respect to observers at infinity) as $\alpha' \to 0$.

The AdS/CFT correspondence is now apparent; identifying the free gravity parts of the two viewpoints, the remaining theories on each side must also be identical. Thus, the 5-dimensional AdS supergeometry is a dual description (in this limit) of the 4-dimensional SYM theory. This correspondence is particularly useful since we can use perturbation theory on the SYM side when $g_{\text{YM}}^2 N \sim g_s N \ll 1$, while the supergravity approximation is valid when $R \sim g_s N \gg 1$. The supergravity theory may be used to calculate non-perturbative properties of SYM.

The above discussion shows how the correspondence takes place in supergravity at low energies and as $\alpha' \to 0$. The full conjecture states that type IIB string theory on an $\text{AdS}_5 \times S^5$ background is dual (for all values of g_s, N and α') to the conformal $\mathcal{N} = 4$ SYM theory in four dimensions.

2.1.3 The general correspondence

The AdS/CFT correspondence as stated above concerns a highly special and non-physical conformal YM theory with $\mathcal{N} = 4$ supersymmetry. To make contact with experiments, the best option would be to be able to describe the Standard Model of particle physics non-perturbatively with a gravitational dual. But also theories with some supersymmetry have a chance of being relevant for observations, e.g. the minimal supersymmetric extension of the Standard Model with $\mathcal{N} = 1$.

From a more mathematical point of view, extending the AdS/CFT correspondence to other YM models may be considered as a first step on the road to a general identification of YM and gravitational theories. In this sense, also extensions to other dimensions than four are interesting.

For correspondences with different dimension, if we replace the D3 brane in the previous section with a Dp brane for any p (using IIA or IIB theory according to whether p is even or odd), we get a correspondence between supergravity on $\mathrm{AdS}_{p+2} \times S^{8-p}$ and SYM in $p+1$ dimensions. In flat space, the SYM theory will always have 16 preserved supercharges, corresponding to breaking half of the supersymmetry of the original type II string theory.

To extend the correspondence to theories with less supersymmetry, one may modify the geometry and topology of the space in which the D-brane exists. Wrapping the D-brane on a Calabi-Yau manifold or an orbifold, or on non-trivial cycles within such spaces breaks more supersymmetries. Paper 1 discusses two such examples, one using *wrapped* branes and another using *fractional* branes.

2.2 Wrapped branes

The near horizon geometry of a Dp-brane is a product of a $p+2$-dimensional anti de Sitter space with a sphere. In supergravity, solutions with this geometry are found by "inverting" a Kaluza-Klein compactification of 10-dimensional (or 11-dimensional) supergravity to $p+3$ dimensions. This is possible since the compactification may be formulated as a *consistent truncation* [30], meaning that any solution of the truncated theory can be "lifted" to a solution of the full 10- or 11-dimensional theory.

It was realized in [17] that the supergravity description of a D-brane wrapped on a nontrivial cycle of a manifold requires a form of "topological twisting" in order to preserve half of the supersymmetry as it should. The geometric explanation for this follows.

The field theory on the $p+1$-dimensional world volume of a Dp-brane consists of a gauge field and $9-p$ scalars. The scalars are remnants of the string movement in the "outside" dimensions, and may be interpreted as degrees of freedom corresponding to the movement of the D-brane itself in the external space. In a nontrivial space, these degrees of freedom are not arbitrary functions of the world-volume. They must be sections of the normal bundle of the D-brane.

On the supergravity side, the truncation (Kaluza-Klein compactification) of the theory to $p+3$ dimensions introduces gauge fields A_μ into the theory. The above geometry means that A_μ is the connection on the normal bundle [31].

2.3 Fractional branes

2.3.1 Orbifolds

Dealing with strings in compact spaces may be technically difficult, since the metric of the background is non-trivial. In fact, in the case of 6-dimensional Calabi-Yau manifolds, not a single example of an explicit metric has been found. However, some simplified versions of such spaces exist.

An *orbifold* is made by taking a smooth space M (e.g. flat space $\mathbb{R}^d$ or $\mathbb{C}^d$), and identifying points transforming into each other under some discrete group Γ. If some points in M are stationary (fixed points) under the transformation induced by a non-trivial element of Γ, the resulting space will be singular at these points and is called an orbifold.

2.3.2 D-branes on orbifolds

To be invariant under the group action, a generic D-brane in the orbifold is represented by n D-branes placed on n points in the covering space $\mathbb{C}^d$ that are identified by Γ. These branes transform in a representation of the group Γ which is not in general irreducible. On a fixed point of the group, i.e., a singular point of the orbifold, they may be reduced to *fractional branes* transforming under irreducible representations of Γ. On an orbifold $\mathbb{C}^d/\mathbb{Z}_n$, the fractional branes will have charge Q/n with respect to the regular brane charge Q.

As an example[32, 33, 34], consider the orbifold $\mathbb{C}^2/\mathbb{Z}_2$, where the non-trivial element of $\mathbb{Z}_2$ acts as $(x,y) \to (-x,-y)$. A D0-brane at a generic point in the orbifold (or a Dp-brane which is point-like when restricted to the orbifold) is represented by two D0-branes in $\mathbb{C}^2$— one at a point (x,y) and another, identical one at $(-x,-y)$. Open string states will have a Chan-Paton factor which is a two by two matrix, labelled by the branes. The action of the non-trivial group element on the Chan-Paton matrix is to interchange the branes, i.e., the regular representation

$$\{1, \sigma_1\}, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.4}$$

This is a reducible representation ($\mathbb{Z}_2$ is an Abelian group, which only has one-dimensional irreducibles). The irreducible representations of $\mathbb{Z}_2$ are $\{1,1\}$ and $\{1,-1\}$, and so we can construct Chan-Paton factors transforming under these, corresponding to a single D-brane. However, the states can not be invariant under the $\mathbb{Z}_2$ action unless the brane is placed on a fixed point of the group. These are the fractional branes on this orbifold.

2.3.3 Connection between wrapped and fractional branes

An orbifold of the type $\mathbb{C}^2/\Gamma$, where Γ is a discrete subgroup of SU(2), can be shown to be a limit of a smooth space called an asymptotically locally Euclidean (ALE) space. In this sort of space, the singular point is replaced (resolved) by a compact manifold (the vanishing cycle). In the limit where the compact manifold has zero volume, the orbifold is recovered. For the case $\mathbb{C}^2/\mathbb{Z}_2$, the compact space is the projective sphere $\mathbb{CP}^1 \cong S^2$.

A fractional D-brane may be seen as a D-brane wrapping the vanishing cycle of an ALE space, e.g. a fractional D2 brane may be seen as a D4 brane with two dimensions wrapped on the vanishing cycle $\mathbb{CP}^1$.

2.4 The Eguchi-Hanson metric

The ALE space asymptotically like $\mathbb{C}^2/\mathbb{Z}_2$ in the example above actually has a well-known metric known as the *Eguchi-Hanson metric* [35]. This is a solution of the Euclidean Einstein equations with a *self-dual* Riemann curvature,

$$R_{\mu\nu\lambda\sigma} = \frac{1}{2}\epsilon_{\mu\nu\omega\rho}R_{\omega\rho\lambda\sigma}. \tag{2.5}$$

The metric may be written

$$ds^2 = \left(1 - \left(\frac{a}{r}\right)^4\right)^{-1} dr^2 + \frac{1}{4}\left(1 - \left(\frac{a}{r}\right)^4\right)(d\psi + \cos\theta d\phi)^2 + \frac{1}{4}r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.6}$$

Defining a new coordinate $u^2 = r^2(1-(\frac{a}{r})^4)$, we find that $u\,du \approx 2r\,dr$ when $r \to a$, so the metric in the u, ψ plane may be written [36]

$$ds^2 \approx \frac{1}{4}\left(du^2 + u^2 d\psi^2\right) \tag{2.7}$$

near the apparent singularity $r = a$. This is known as a "bolt", which is not singular as long as ψ runs from 0 to 2π (otherwise, there will be a conic singularity at $r = a$).

In our case the original metric, eq. (2.6) looks like it describes a S^3 at infinity, but this would mean that the range of ψ was $0 \leq \psi < 4\pi$. Restricting to a period of 2π means that the asymptotic topology is $S^3/\mathbb{Z}_2 \cong \mathbb{RP}^3$. This is the correct asymptotics for the ALE space. Letting $a \to 0$, the metric becomes flat everywhere except for a singularity at the origin. This is exactly the metric for $\mathbb{C}^2/\mathbb{Z}_2$. The Eguchi-Hanson metric thus describes a smooth space which asymptotes to $\mathbb{C}^2/\mathbb{Z}_2$, and the limit of vanishing volume of the $\mathbb{CP}^1$ cycle is taken by letting $a \to 0$.

2.5 Probing the moduli space of gauge theories

The low-energy limit of a configuration of N Dp-branes contains a p-dimensional SU(N) Yang-Mills theory breaking to SU($N-1$) with a Higgs vacuum expectation value corresponding to the separation of the branes. Since the Higgs vev is typically used as a parameter of the moduli space of a YM theory, this space can be probed by tuning the relative position of the D-branes.

The procedure used for this probing is to let a "probe brane" move in the geometry created by $N-1$ branes, in the supergravity low-energy limit. The effective action of the low-energy theory is the DBI action, eq. (1.45), plus a Wess-Zumino part, eq. (1.47),

$$S = -T_p \int d^{p+1}x e^{-\Phi} \det \sqrt{G_{\mu\nu} + B_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}} + T_p \int C_{(p)} \wedge e^{2\pi\alpha' F + B}, \tag{2.8}$$

and the probing is achieved by inserting the (induced) metric of $N-1$ branes into this action.

For the case of a Dp-brane in flat space, the metric is given by eq. (2.2), and inserting this into eq. (2.8) gives

$$S \approx -T_p \int d^{p+1}x \left(\frac{1}{2}\partial^\mu x^i \partial_\mu x_i + (2\pi\alpha')^2 F^{\mu\nu} F_{\mu\nu} \right) + \text{const.}, \tag{2.9}$$

which, if we identify $\Phi^i = x^i/(4\pi\alpha')$ as the gauge theory scalars, is exactly (apart from the irrelevant constant term) the action of SYM theory in $p+1$ dimensions with 16 supercharges. The metric on the moduli space in this case is flat, as a cause of the high amount of supersymmetry in this case. To get more interesting metrics, one should look at configurations breaking supersymmetry. This is what we do in paper 1.

In the paper, we look at two geometries corresponding to 8 supercharges in 2+1 dimensions ($\mathcal{N} = 4$ SUSY). The supersymmetry is broken from the flat case of 16 charges in two different ways. In the first case, we look at a D4-brane wrapped on a two-cycle in an ALE space, topologically equivalent to the Eguchi-Hanson space above. In the second case, we consider fractional branes on a $\mathbb{C}^2/\mathbb{Z}_2$ orbifold. Both geometries lead to the same SYM theory, as expected since the two cases are related, as we have shown above.

The supergravity solution corresponding to the N wrapped branes was found by a slight detour. We start with a solution of 7-dimensional gauged supergravity from [31] corresponding to an M5 brane in 11-dimensional supergravity. The solution is uplifted to 11 dimensions, and then one dimension tangential to the brane is compactified to obtain a D4 brane in 10-dimensional supergravity. The final metric is found to have a form which may be described as a "warped" Eguchi-Hanson space, as expected from this topology.

In the fractional case, we actually consider a solution corresponding to N fractional D2 branes and M D6 branes (the D6 branes wrapping the whole

ALE space in addition to two flat spatial dimensions common to the D2 and D6 branes). The D6 branes contribute M hypermultiplets to the SYM theory transforming in the fundamental representation of the gauge group SU(N). In this case, the solution is found by considering the branes as "wrapped" on the zero-volume vanishing 2-cycle of the ALE space. On this cycle there is a constant background B-field necessary to define a sensible CFT for this geometry [37, 38].

The metrics on the moduli space in the two cases is computed to

$$ds^2_{\text{wrap}} = \frac{1}{g^2_{\text{YM}}(\mu)}\left(d\mu^2 + \mu^2 d\Omega^2\right) + g^2_{\text{YM}}(\mu)\left(d\Sigma + \frac{N\cos\theta}{4\pi}d\varphi\right)^2 , \quad (2.10)$$

$$ds^2_{\text{frac}} = \frac{1}{g^2_{\text{YM}}(\mu)}\left(d\mu^2 + \mu^2 d\Omega^2\right) + g^2_{\text{YM}}(\mu)\left(d\Sigma + \frac{(2N-M)\cos\theta}{8\pi}d\varphi\right)^2 , \quad (2.11)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$, and μ, θ, ψ, and Σ are scalars of the theory. This corresponds to the *perturbative* result for $d = 2+1$, $\mathcal{N} = 4$ SYM theory. The reason why we are not able to find the total non-perturbative result seems to be that an enhançon mechanism [39] is taking place. The running volume of the vanishing 2-cycle of the ALE space goes to zero at a certain locus, signalling that the supergravity description breaks down at this point.

Chapter 3

Stability, topology and geometry

3.1 Sheaves, D-branes and stability

The field strength of a Yang-Mills theory defined on a manifold X defines a *vector bundle* on the manifold. Vector bundles are of great interest in differential and algebraic geometry, as well as a "generalisation", a *coherent sheaf*. Roughly speaking, a coherent sheaf is a vector bundle with non-constant rank. More formally, a sheaf E is defined as an association of a group (or vector space) $E(U)$ to each open subset $U \subset X$, which is called the *sections* of E over U, together with a *restriction map* for all open subsets $V \subset U$ which must satisfy certain conditions. A collection of D-branes on a space X may be approximated by a sheaf on X in the limit where X is large enough to make stringy corrections irrelevant. More properly, a D-brane should be considered as an element of the "derived category of coherent sheaves" $D(X)$.

It is of interest in geometry to classify the possible sheaves on a manifold (or variety). This classification is made simpler by introducing notions of *stability* which will be discussed below. Classification of *stable* sheaves turn out to be an easier task than the full classification problem, and in a sense an unstable sheaf may be "deconstructed" into stable sheaves. The concept of stability was introduced by Mumford, in the case of vector bundles on curves.

In physics, there is also a natural condition of stability; a D-brane is stable if it can not decay into other D-branes. Perhaps surprisingly, there seems to be a connection between these two types of stability. A consequence of this connection is studied in paper 2, where we find that a modification of the traditional stability condition used in geometry is necessary to describe physically stable D-branes.

More on stability, vector bundles, and sheaves may be found in [40].

3.2 Mumford and Gieseker stability

The geometry of B-type D-branes was discussed in section 1.6.3. The conditions for preserving supersymmetry was that the vector bundle, E, was holomorphic, and that the field strength satisfied an equation. To lowest order the equation was given in eq. (1.65),

$$g^{\mu\bar{\nu}}F_{\mu\bar{\nu}} = 0. \tag{3.1}$$

This equation is a special case of the *Hermitian Yang-Mills* (also known as Hermitian-Einstein) equation

$$g^{\mu\bar{\nu}}F_{\mu\bar{\nu}} = \mu I, \tag{3.2}$$

where I is an identity matrix. By the Donaldson-Uhlenbeck-Yau (DUY) theorem, the existence of a solution to this equation is equivalent to *Mumford stability* of the vector bundle E. Taking the Hodge dual of the above equation gives

$$F \wedge \frac{\omega^n}{(n-1)!} = \mu I \frac{\omega^n}{n!}. \tag{3.3}$$

We have introduced the Kähler form $\omega_{\mu\bar{\nu}} = g_{\mu\bar{\nu}}$ of the target space with the property that $\omega^n/n!$ is the volume form[1].

Mumford stability, or μ-stability (more properly named Mumford-Takemoto stability for the case of general dimension of the base manifold) is a condition for a vector bundle or sheaf which requires the definition of the *slope* (or *normalised degree*) μ,

$$\mu(E) = \frac{1}{\operatorname{rk} E}\int c_1(E)\wedge\omega^{n-1} = \frac{\deg E}{\operatorname{rk} E}. \tag{3.4}$$

This stability condition does not really need a Kähler manifold, any ample line bundle could be used in place of ω, but we will only use it in the case where the base manifold is Kähler.

Definition 3.2.1 *Let E be a torsion free coherent sheaf on a Kähler manifold. E is* Mumford stable *(*semistable*) if, for all coherent subsheaves E' of E with $0 < \operatorname{rk} E' < \operatorname{rk} E$, we have*

$$\mu(E') < \mu(E) \qquad (\mu(E') \leq \mu(E)). \tag{3.5}$$

E is called unstable *if it is not stable and* strictly semistable *if it is semistable but not stable.*

The connection between Mumford stability and equation (3.3) was proved by Donaldson[41] for the case where X is a (complex) surface, and by Uhlenbeck and Yau[42] for the general case.

[1] There is an unfortunate misprint in eq. (4) and the unnumbered last equation on p. 76 of paper 2. The volume form is missing from the right side of these equations.

Theorem 3.2.2 (Donaldson, Uhlenbeck, Yau) *Let E be a holomorphic vector bundle on a compact Kähler manifold X with Kähler form ω. If there exists a Hermitian-Einstein connection on E, then E is Mumford stable. Conversely, if E is Mumford stable, there exists a Hermitian-Einstein connection on E and this connection is unique up to (C^∞) automorphisms of E.*

Through the DUY theorem we therefore get a connection between the "topological" condition of (Mumford) stability and the geometric problem of finding solutions of (3.3), and thus also the problem of finding supersymmetric (BPS) solutions of Yang-Mills theory.

The definition of Mumford-Takemoto stability is not the best suited in higher dimensions. A better behaved condition in many cases is known as *Gieseker stability.* A correspondence to a non-linear differential equation, similar to the DUY theorem, has been studied by Leung [43].

3.3 Π-Stability

In a theory with extended supersymmetry, there exist a set of bosonic symmetry generators commuting with all the other generators of the symmetry algebra. In an $\mathcal{N} = 2$ supersymmetric theory there is a single central charge, denoted $\mathcal{Z}$. All states in the theory satisfy the *Bogomol'nyi-Prasad-Sommerfield (BPS) bound*,

$$m \geq |\mathcal{Z}|, \tag{3.6}$$

where m is the mass and $\mathcal{Z}$ is the central charge of the state in question. The states saturating this bound, are exactly the BPS states breaking one half of the supersymmetry, of which D-branes are examples.

The mass of a D-brane E is thus given by the central charge, which again is given by the R–R charge of the D-brane,

$$m(E) = |\mathcal{Z}(Q(E))|. \tag{3.7}$$

By conservation of the central charge and energy, a state can only decay into states with the same phase of $\mathcal{Z}$:

$$\varphi(E') = \varphi(E) := \frac{1}{\pi} \operatorname{Im} \log \mathcal{Z}(Q(E)) \tag{3.8}$$

when E' is a (possible) decay product of E.

This motivated the definition of Π-stability [44], modelled after Mumford stability as follows: *A coherent sheaf E is* Π*-stable if, for all subsheaves E' one has*

$$\varphi(E') < \varphi(E). \tag{3.9}$$

The central charge $\mathcal{Z}(E)$ may be calculated from the topology of a B-type brane by using the representation of the R–R-charge in terms of the *Mukai vector*

$$Q(E) = \mathrm{ch}(E)\sqrt{\mathrm{Td}(X)}, \tag{3.10}$$

where $\mathrm{Td}(X)$ is the *Todd class* of the base CY-manifold X. The central charge is then given by [45]

$$Z(E) = -\int_X e^{-i\omega} Q(E) = -\int_X e^{-i\omega}\,\mathrm{ch}(E)\sqrt{\mathrm{Td}(X)}, \tag{3.11}$$

where ω is the complexified Kähler form of X.

3.3.1 Example: The Quintic

On the Quintic, the Kähler moduli space is one-dimensional, and we parametrise the Kähler form as $\omega = tJ$, where J is the cohomology class of the hyperplane bundle and $t = B + iV$. We also parametrise the Chern character of a sheaf as[2]

$$\mathrm{ch}(E) = r + \frac{Ch_1}{3}J + \frac{Ch_2}{6}J^2 + \frac{Ch_3}{6}J^3. \tag{3.12}$$

The Todd class for the Quintic is $\mathrm{Td}(X) = 1 + \frac{5}{6}J^2$. With $B = 0$, the expression for $Z(E)$ above reduces to

$$Z(E) = -5irt^3 + 5Ch_1t^2 + \left(\frac{25}{2}ir + 5iCh_2\right)t - \frac{25}{6}Ch_1 - 5Ch_3. \tag{3.13}$$

The requirement (3.9) for Π-stability may be expressed as

$$\operatorname{Im} Z' \operatorname{Re} Z - \operatorname{Im} Z \operatorname{Re} Z' < 0, \tag{3.14}$$

which in this case becomes

$$\begin{aligned}
&\left(\frac{Ch_1'}{r'} - \frac{Ch_1}{r}\right)t^5 + \\
&\quad\left(\frac{10}{3}\frac{Ch_1}{r} - \frac{10}{3}\frac{Ch_1'}{r'} + \frac{Ch_2'Ch_1}{rr'} - \frac{Ch_2Ch_1'}{rr'} + \frac{Ch_3}{r} - \frac{Ch_3'}{r'}\right)t^3 + \\
&\quad\quad\left(\frac{25}{12}\frac{Ch_1'}{r'} - \frac{25}{12}\frac{Ch_1}{r} + \frac{5}{2}\frac{Ch_3'}{r'} - \frac{5}{2}\frac{Ch_3}{r} + \frac{5}{6}\frac{Ch_2Ch_1'}{rr'}\right. \\
&\quad\quad\quad\left. - \frac{5}{6}\frac{Ch_2'Ch_1}{rr'} + \frac{Ch_2Ch_3'}{rr'} - \frac{Ch_2'Ch_3}{rr'}\right)t < 0
\end{aligned} \tag{3.15}$$

[2] This parametrisation is done to make contact with the *generalised slopes* $\mu^{(k)} := Ch_k/r$ defined in paper 2.

Thus, the requirement for a sheaf to be Π-stable on the quintic in the limit $t \to \infty$ is that, for all $E' \subset E$,

$$\frac{Ch_1'}{r'} \leq \frac{Ch_1}{r}, \tag{3.16}$$

and, if $Ch_1/r = Ch_1'/r'$ for some $E' \subset E$,

$$\frac{Ch_1}{r}\frac{Ch_2'}{r'} - \frac{Ch_3'}{r'} \leq \frac{Ch_1'}{r'}\frac{Ch_2}{r} - \frac{Ch_3}{r}, \tag{3.17}$$

and, if also this inequality is satisfied as an equality for some E',

$$\frac{5}{2}\frac{Ch_3'}{r'} + \frac{5}{6}\frac{Ch_2 Ch_1'}{rr'} + \frac{Ch_2 Ch_3'}{rr'} < \frac{5}{2}\frac{Ch_3}{r} + \frac{5}{6}\frac{Ch_2' Ch_1}{rr'} + \frac{Ch_2' Ch_3}{rr'} \tag{3.18}$$

If strictly satisfied, the first inequality of this series implies that the sheaf is Mumford stable. If satisfied as an equality for some E', this implies Mumford semistability. We can therefore see that all Mumford stable sheaves are Π-stable, and all Π-stable sheaves are Mumford semistable. However, not all Π-stable sheaves are Mumford stable.

3.4 Geometric Π-stability

The "geometric limit" of a D-brane is not exactly the Yang-Mills theory, but includes corrections. The equation for the gauge field strength found by Mariño et. al [26] for the case of a D6-brane wrapping a 3-fold with $B = 0$ is given in eq. (1.66)

$$\omega \wedge \omega \wedge F - \frac{1}{3}(F \wedge F \wedge F) = \cot\theta_E(\omega \wedge F \wedge F - \frac{1}{3}\omega \wedge \omega \wedge \omega), \tag{3.19}$$

where ω is the Kähler form and θ_E is a constant determined by the topology of the gauge vector bundle E. The value of $\cot\theta_E$ may be found by integrating equation (3.19).

Since equation (3.19) contains string corrections with respect to the Hermitian Einstein equation, and Π-stability is proposed as a string corrected stability condition, one might expect that there is a connection between these two. In paper 2 we consider the geometric limit of Π-stability, which we call π-stability, and a slight deformation of eq. (3.19). By building on a result by Leung [43], we find that π-stability is the condition for eq. (3.19) (modified to take into account the factor $\sqrt{\hat{\mathcal{A}}(X)}$ in eq. (1.48)) to have a unique solution.

Chapter 4

Gepner models, D-branes and algebraic geometry

4.1 Boundary states in conformal field theory

4.1.1 D-branes as boundary states and rational CFT

The description of D-branes used previously in this thesis has been mostly from the space-time point of view. We have for the most part described Dp-branes as solutions of supergravity corresponding to p-dimensional extended objects carrying R–R charge, although they were derived as objects to which open strings can end. From the string point of view, the branes should have a description in terms of conformal field theory. The correct description is a *boundary state*, which will be reviewed below.

There are special conformal field theories which are exactly solvable, known as rational CFTs. One such class of theories are the *minimal models* with central charge $c < 3$ (we are here considering $\mathcal{N} = 2$ supersymmetric minimal models, for non-supersymmetric models the minimal models have $c < 1$). The existence and exactness of such models goes back to Kac. In $\mathcal{N} = 2$ models with $c < 3$ there are only a finite number of allowed highest weight states, and this simplifies the theory significantly. Tensor products of minimal models such that the total CFT has $c = 9$ was studied by Gepner [46] as an alternative to the compactification of the extra 6 dimensions of superstring theory. From the CFT point of view, the constraint that the total central charge (including ghost contributions) should be zero is satisfied with this setup equally well as the nonlinear sigma model.

Combining the Gepner construction and the boundary state approach, we are led to the spectrum of boundary states in a Gepner model, first studied by Recknagel and Schomerus [47] and interpreted geometrically for the case of the Quintic in [48]. In paper 3, we are studying an extension to the original "Recknagel-Schomerus branes" by generalising the boundary conditions, the so-called "permutation branes" [49]. D-branes in Gepner models have an interpretation in terms of an algebraic construction called *matrix factorisations* [27]. We find that the permutation branes correspond to the specific class of *linear* matrix factorisations.

4.1.2 Ishibashi states

Open strings may be described perturbatively by a two-dimensional conformal field theory (CFT) on the upper half-plane, known as boundary CFT (BCFT). The boundary conditions for $\sigma = 0, \pi$ may be conformally transformed into a boundary condition for $z \in \mathbb{R}$, in term of a complex coordinate z. One must require that there is no energy flows across the boundary, i.e. across the real line. This gives the condition $T(z) = \bar{T}(\bar{z})$ for $z = \bar{z}$.

For an extended symmetry algebra $\mathcal{W}$ with an additional generator $W(z)$, e.g., generating supersymmetry, the boundary condition may be more general:

$$W(z) = \Omega(\bar{W})(\bar{z}) \qquad z = \bar{z}, \tag{4.1}$$

where Ω is an automorphism of the algebra acting trivially on the energy momentum tensor.

In the CFT, one way of implementing the boundary conditions is the *boundary state approach* [50, 51, 52] (See e.g. [53] for a review). With the conformal transformation (different from the one considered above) $(\tau, \sigma) \mapsto (-\sigma, \tau)$ the boundary in the "space" coordinate σ is mapped onto a boundary in "time", $\tau = 0, \pi$. This is how the open-closed duality appears in CFT. A one-loop diagram describing an open string stretching between two D-branes is mapped onto a closed-string diagram where the string propagates from an initial state to a final state, these states given by the original boundary conditions at the two branes.

The boundary conditions in eq. (4.1) may be implemented in the *bulk* CFT by including *Ishibashi states* $|i\rangle\rangle_\Omega$ obeying the *gluing conditions*

$$\left(L_n - \bar{L}_{-n}\right) |i\rangle\rangle_\Omega = 0, \tag{4.2}$$

$$\left(W_n - (-1)^{h_W} \Omega(\bar{W}_{-n})\right) |i\rangle\rangle_\Omega = 0. \tag{4.3}$$

Such a state may be represented by the formal expression

$$|i\rangle\rangle_\Omega = \sum_{N=0}^{\infty} |i, N\rangle \otimes V_\Omega U \, |i, N\rangle \, , \tag{4.4}$$

where the sum is over an orthonormal base of energy eigenstates in the Hilbert space associated with the highest weight state $|i, 0\rangle$. There is thus an Ishibashi state for each highest weight state of the bulk theory. A general "boundary state" may be created as a linear combination of Ishibashi states,

$$\|i\rangle\rangle = \sum_j B^i{}_j |j\rangle\rangle. \tag{4.5}$$

A true boundary state must generate a modular invariant partition function, which gives constraints on the coefficients ${B^i}_j$. Cardy[54] found that a boundary state may be created by coefficients of the modular S-matrix,

$$ {B^i}_j = \frac{S_{ij}}{S_{0j}}. \tag{4.6} $$

4.1.3 Boundary states in SCFT

For an $\mathcal{N} = 2$ supersymmetric CFT the gluing conditions (4.1) applies to the supersymmetry generators $G^{\pm}$ and the $U(1)$ current J. The Ishibashi states that preserve spacetime $\mathcal{N} = 1$ supersymmetry may therefore be grouped into A_θ-type and B-type states[47, 55], labelled after which of the boundary conditions of section 1.6.1 they correspond to.

4.2 The Gepner construction

4.2.1 Compactification and CFT

Gepner [46] introduced a generalisation of the standard compactification of the extra dimensions of string theory by replacing the $10-D$ free superfields by an "internal CFT" having central charge $15 - \frac{3}{2}D$. The idea is that any CFT with the correct central charge (such that $c = 0$ altogether), leads to Weyl invariance, and thus may be used in place of the usual sigma model in the internal sector. The construction Gepner used was a tensor product of so-called *minimal models*, which are special "rational" CFTs which allow one to calculate all correlation functions exactly, i.e., not using perturbation theory. Although the construction itself is highly non-geometrical, it was found that the Gepner models nevertheless admit an interpretation as a (very special) point in the moduli space of a geometrical model.

4.2.2 Minimal models

A $\mathcal{N} = 2$ minimal model is a superconformal field theory with central charge $c < 3$. Such a model is particularly simple, since it has only a finite number of highest weight states. The minimal models are characterised by an integer $k = 1, 2, \ldots$, and model k has central charge $c = \frac{3k}{k+2}$. A HWS in such a model is characterised by three integers l, m, s such that

$$ \begin{array}{cc} l = 0, 1, \ldots, k & s = 0, 1, 2, 3 \\ m = 0, 1, \ldots, 2k+4 & l + m + s \in 2\mathbb{Z} \end{array} \tag{4.7} $$

The state (l, m, s) is identified with the state $(k - l, m + k + 2, s + 2)$ (*field identification*).

4.2.3 Gepner product

A state in the Gepner model is a product of minimal model states subjected to a GSO projection. The fermion number is related to the $U(1)$ charge, and the GSO projection amounts to keeping states with odd charge[1] To make the projection consistent, one must also introduce *twisted states* into the partition function. This ensures that the partition function is modular invariant.

A product of r minimal models has highest weight states labelled as

$$\lambda := (l_1, \ldots, l_r) \qquad \mu := (s_0; m_1, \ldots, m_r; s_1, \ldots, s_r), \tag{4.8}$$

where s_0 labels the spin of the "external" fermions corresponding to the flat dimensions. In the Gepner construction, an additional projection must be made onto states that are pure Neveu-Schwarz or pure Ramond states, i.e., all states in the product must be either NS or R. The total projection may be formulated by introducing vectors β_0, β_j in "μ-space", with products

$$2\beta_0 \cdot \mu = -\frac{d}{2}\frac{s_0}{2} - \sum_{j=1}^{r} \frac{s_j}{2} + \sum_{j=1}^{r} \frac{m_j}{k_j+2}, \tag{4.9}$$

$$\beta_j \cdot \mu = -\frac{d}{2}\frac{s_0}{2} - \frac{s_j}{2}. \tag{4.10}$$

Where we have left the dimension d of the external space arbitrary. Projections onto states with $2\beta_0 \cdot \mu$ odd and $\beta_j \cdot \mu$ integer will implement the Gepner projection.

4.2.4 Boundary states in Gepner models

Ishibashi states may be created in a Gepner Model by the same construction as discussed in section 4.1.2. However, because of the Gepner projection, it is not obvious that boundary states (with modular invariant partition function) may be created simply by using the S-matrix as in eq. (4.6).

D-branes in Gepner models were first studied by Recknagel and Schomerus [47] who found that the boundary coefficients (up to normalisation) for non-singular Gepner models are given by

$$B_\alpha^{\lambda,\mu} = (-1)^{s_0^2/2} e^{-i\pi(d/2)(s_0 S_0/2)} \prod_{j=1}^{r} \frac{\sin \frac{\pi(l_j+1)(L_j+1)}{k_j+2}}{\sin^{1/2} \frac{\pi(l_j+1)}{k_j+2}} e^{i\pi m_j M_j/(k_j+2)} e^{-i\pi s_j S_j/2}, \tag{4.11}$$

where the labels L_j, M_j, S_0, S_j are labels in the same ranges as l_j, m_j, s_0, s_j above.

[1]The projection may be realized as a *simple current extension*[56, 22]. This is a more general procedure for constructing a modular invariant partition function, and will in this case give the same results as the original "β-method" introduced by Gepner.

4.3 Connection to geometry

4.3.1 The Gepner model and geometry

The Gepner construction has been shown to be more strongly connected to ordinary Calabi-Yau compactification than one might guess from the construction itself. A Gepner product of $r = n + 2$ minimal models k_i, $i = 1, \ldots, r$ with total central charge $c = \sum_{i=1}^{r} \frac{3k_i}{k_i+2} = 3n$ is in fact a special "small radius" point in the parameter space of closed string theory compactified on an n-dimensional Calabi-Yau space constructed as the zero locus of the polynomial

$$W(x) = \sum_i (x_i)^{k_i+2} \qquad (4.12)$$

in the weighted projective space $W\mathbb{P}^{n+1}_{(w_1,w_2,\ldots,w_r)}$, where $w_i = \frac{K}{k_i+2}$ and K is the least common multiple of k_i+2. These facts were first realized in [57, 58] and later formalised in Witten's construction of the linear sigma model [59] interpolating between the "large radius" point in the parameter space, which may be studied with ordinary algebraic geometry, and the Gepner point.

This correspondence is also valid for $r = n + 1$, where one may add a "trivial" extra minimal model $k_r = 0$ and use the same formalism as above.

A Gepner model may be interpreted as the IR fixed point of a $\mathcal{N} = 2$ *Landau-Ginzburg model* [60] defined by a superspace action

$$S = \int d^2zd^4\theta K(\Phi, \bar{\Phi}) + \left(\int d^2zd^2\theta W(\Phi) + c.c. \right), \qquad (4.13)$$

where K is an irrelevant kinetic term and the superpotential W is defined by the same polynomial as in eq. (4.12) (See [61] for a review).

4.3.2 D-branes and algebraic geometry

By a proposal of Kontsevich, a (B-type) D-brane in a Landau-Ginzburg model was in [27] interpreted in terms of a construction in algebraic geometry known as a *matrix factorisation* of the superpotential $W(x)$. This is a pair of matrices (p_0, p_1) such that

$$p_0p_1 = p_1p_0 = W\mathrm{id}, \qquad (4.14)$$

where id is the identity matrix. The correspondence has been further studied in [62, 63, 64, 65, 66].

A D-brane in a LG model is described by defining the model on a manifold with a boundary. The immediate problem with this description is that this model is no longer BRST invariant. A B-type supersymmetry transformation, which left the model with no boundary invariant, of the action leaves a boundary term which may be written

$$\delta S = \frac{i}{2} \int dx^0 \left(\epsilon\bar{\eta}\bar{W}' + \bar{\epsilon}\eta W' \right), \qquad (4.15)$$

where $\epsilon, \bar{\epsilon}$ are (Grassmann) parameters of the transformation and $\eta, \bar{\eta}$ are fermions of the theory. This is the Warner problem [67]. To solve the problem, one must introduce a boundary (super)field Π satisfying $D\Pi = E(\Phi)$, with a (boundary) action reading

$$S_{\delta\Sigma} = -\frac{1}{2}\int dx^0 d^2\theta \bar{\Pi}\Pi\Big|_0^\pi - \frac{i}{2}\int dx^0 d\theta \Pi J(\Phi)_{\bar{\theta}=0}\Big|_0^\pi + c.c.. \tag{4.16}$$

The boundary potentials E and J are determined by the supersymmetry transformation of eq. (4.16), which becomes

$$\delta S_{\delta\Sigma} = -\frac{i}{2}\int dx^0 \left(\epsilon\bar{\eta}(\bar{E}\bar{J})' + \bar{\epsilon}\eta(EJ)'\right), \tag{4.17}$$

showing that by choosing them such that $EJ = W$, the Warner problem is solved.

The realisation of Kontsevich, Kapustin and Li was that the above problem corresponds exactly to finding matrix factorisations of W. This is a method found by Eisenbud [68] of describing an algebraic module, which again leads to the construction of a sheaf. It is then natural to believe that the construction in the LG model should lead to a D-brane corresponding in some way to the sheaf defined in this way from algebraic geometry.

4.4 Modules and resolutions

In algebraic geometry, a matrix factorisation defines a sheaf on the manifold X through the complex

$$\cdots \xrightarrow{p_1} P_0 \xrightarrow{p_0} P_1 \xrightarrow{p_1} P_0 \longrightarrow P \longrightarrow 0, \tag{4.18}$$

where P_1, P_0 are free R-modules and R is the coordinate ring of the manifold X. A module P is defined by the above complex, which is a *free resolution* of the module, and a sheaf $\widetilde{P}$ is defined by *sheafification* of P. Many properties of the sheaves may be obtained from this algebraic description. Some general properties of modules and free resolutions will be reviewed below.

The definition of an R-module P, where R is a ring, is identical to that of a k-vectorspace (where k is a field such as $\mathbb{R}$ or $\mathbb{C}$), only with R replacing the field k. The difference between a ring and a field is that there may be elements $f \in R$ which have no inverse. Thus, the equation $fw = 0$, with $0 \neq f \in R$ and $w \in P$ does not necessarily imply $w = 0$.

The simplest kind of module is the one most similar to an n-dimensional vector space, the *free* module R^n. This is defined, in parallel to the vector space $\mathbb{C}^n$, by taking the direct sum of n copies of R. In terms of a basis $\{e_i\}_{i=1,\ldots,n}$, we may write $R^n = \sum_{i=1}^n Re_i$.

In a general, finitely generated R-module P, there will be linear relations between a minimal set of generators $\{e_i\}_{i=1,\ldots,n}$, as a result of the non-invertible elements of the coefficient ring R. These relations, say that there are m of them, may be written

$$\sum_{i=1}^{n} f_i^{(j)} e_i = 0, \qquad j = 1, \ldots, m. \tag{4.19}$$

The $n \times k$-matrix $p_1 = (f_i^{(j)})_{ij}$ is called the *presentation matrix* of P. P may now be written as a quotient $P = R^n/\mathrm{image}(p_1)$. The matrix p_1 thus fits in an exact sequence

$$R^m \xrightarrow{p_1} R^n \longrightarrow P \longrightarrow 0. \tag{4.20}$$

This is the first step of a *free resolution* of P. The image of p_1, which is also a R-module, may be considered to be generated by a basis for R^m, $\{e'_i\}_{i=1,\ldots,m}$. But there will in general be linear relations between the image of these basis elements, caused by linear relations between the equations (4.19). These relations generate the kernel of the map p_1, which may be described by a presentation matrix p_2 fitting in the exact sequence as

$$R^k \xrightarrow{p_2} R^m \xrightarrow{p_1} R^n \longrightarrow P \longrightarrow 0. \tag{4.21}$$

This process may be repeated again for the kernel of p_2, and so on, yielding a free resolution of the module P. For every finitely generated module over $\mathbb{C}[x_1, \ldots, x_n]$, one can find a finite free resolution, but for modules over a more general ring, there are often no finite ones.

4.5 Permutation branes

4.5.1 Permutation gluing

In a tensor product CFT, such as the Gepner construction, there is a natural family of non-trivial choices for the gluing automorphism Ω of the symmetry algebra $\mathcal{W}$ in eq. (4.1): For a tensor product of n equal factors, choose a permutation $\pi \in S_n$ and let the gluing automorphism act as

$$\Omega_\pi : W^{[k]}(z) \mapsto W^{[\pi(k)]}(z) \tag{4.22}$$

on a $\mathcal{W}$-generator $W^{[k]}(z) := 1 \otimes \cdots \otimes W(z) \otimes \cdots \otimes 1$ with $W(z)$ in the kth factor, for $k = 1, \ldots, n$. The Ishibashi states in eq. (4.4) now become

$$|i\rangle\rangle_\pi = \sum_{\{N\}} |i_1, N_1\rangle \otimes \cdots \otimes |i_n, N_n\rangle \otimes U \left|i_{\pi^{-1}(1)}, N_{\pi^{-1}(1)}\right\rangle \otimes \cdots \otimes \left|i_{\pi^{-1}(n)}, N_{\pi^{-1}(n)}\right\rangle, \tag{4.23}$$

where the sum is over the Hilbert space energy eigenstates of each individual factor.

Such permutation gluing conditions were introduced in [49] for a Gepner model, and the corresponding boundary states for the Quintic were worked out explicitly. In paper 3, we use a more general approach for calculating the boundary states and find the boundary coefficients for permutation branes in general.

The calculation is based on the fact that to obtain fractional branes (under the orbifold action of the Gepner projection) one must start with a boundary state which is invariant under the orbifold group. To obtain such a state, one must add together all boundary states obtained by twisting with the elements of the group. There is a standard expression for the coefficients of such twisted boundary states [69] which we adapt to the case of permutation gluing conditions.

4.5.2 Permutation branes and algebraic geometry

In paper 3, we propose that permutation branes are related to a special class of matrix factorisations called *linear* in [70]. The proposal is based on calculation of open string spectra in CFT and calculation of the corresponding algebraic structures, the Ext-groups.

In general, a D-brane corresponding to a matrix factorisation (p_0, p_1) may be represented with a $\mathbb{Z}_2$-graded complex

$$P_1 \underset{p_0}{\overset{p_1}{\rightleftarrows}} P_0. \tag{4.24}$$

Massless open string states between the D-branes are identified as maps between such complexes [64], which may be grouped into *bosonic* maps $P_i \to P_i$ with degree 0 and *fermionic maps* $P_i \to P_{1-i}$ with degree 1. The maps corresponding to states in the topological field theory again form a $\mathbb{Z}_2$ graded complex

$$\mathbb{H}\mathrm{om}(P, Q) = \bigoplus_{i,j=0,1} \mathrm{Hom}(P_i, Q_j), \tag{4.25}$$

with grading given by $(i + j) \mod 2$. The differential of the complex is the BRST operator D, acting on a homogeneous element of degree k as

$$D\Phi = (q_1 \oplus q_0) \circ \Phi - (-1)^k \Phi \circ (p_1 \oplus p_0), \tag{4.26}$$

where k is the degree of the map Φ. The operator D represents the BRST operator in the CFT. The cohomology of the complex of maps correspond to the Ext-groups of the modules defined by $\operatorname{coker} p_1$ and $\operatorname{coker} q_1$.

Calculating the cohomology of the above complex (4.25) for the linear matrix factorisations, we can compare it to the corresponding spectrum of chiral primary states in the BCFT. In particular, we may compare the *Witten index* (or intersection form), which is equivalent to the Euler number in geometry,

$$\langle\langle \alpha_P \| (-1)^F \| \alpha_Q \rangle\rangle = \chi(P,Q) = \dim H^0 \mathbb{H}\mathrm{om}(P,Q) - \dim H^1 \mathbb{H}\mathrm{om}(P,Q), \tag{4.27}$$

where the boundary states $\|\alpha_{P,Q}\rangle\rangle$ correspond to the complexes P and Q.

Unfortunately, we are not able to prove the correspondence rigorously, and must resort to circumstantial evidence for our claim. Mainly, this is in the form of Ext-groups of the modules calculated with the help of the computer algebra program Macaulay2 [71]. Macaulay2 does exact calculations for explicit examples, and we have created code, included in the appendix of the paper, which produces the rings and modules for a number of the linear matrix factorisations. As far as we have been able to check, the correspondence holds, however, it would be of great interest to find a rigorous proof for this relation. Such a proof would likely be enlightening for the general correspondence between matrix factorisations in algebraic geometry and in Landau-Ginzburg models as well.

Appendix A

Mathematical Concepts

A.1 Complex and algebraic geometry

In this appendix we have gathered some definitions and theorems from topology, algebra, and geometry which are used in the text. The appendix is intended as a reference only, for an introduction to the subjects we refer to [72, 73, 74, 75].

A.1.1 Manifolds and varieties

An n-dimensional *differentiable manifold* M is a smooth space locally homeomorphic to $\mathbb{R}^n$. More formally, such a manifold may be covered with open patches U_i and there is a homeomorphism ϕ_i from each of the U_i to an open subset of $\mathbb{R}^n$. There now exists a map between overlapping coordinate patches $\psi_{ij} = \phi_i \phi_j^{-1}$, and we require this map to be infinitely differentiable. The homeomorphisms give local coordinate systems on the manifold.

In the case where n is even, it may be possible to use complex coordinates on the manifold instead of real coordinates. The is useful if the manifold allows a *complex structure*, which means that one can define the local homeomorphisms ϕ_i such that the transition functions ψ_{ij} are holomorphic functions. A manifold may allow several non-equivalent complex structures. Between two complex manifolds one may also be able to define a *holomorphic map*, which is a map $f : M \to M'$ such that the corresponding maps between open subsets of $\mathbb{C}^n, \mathbb{C}^m$ (where n is the dimension of M and m that of M') are holomorphic.

Projective spaces are the basis of most algebraic geometry. They are studied, in place of the *affine* spaces $\mathbb{C}^n$, because they are *compact*, and thus easier to work with. There are several equivalent definitions of a projective space. One of them is the following: In $\mathbb{C}^{n+1} \setminus \{(0, \ldots, 0)\}$, identify points that differ in a common (complex) factor, i.e.

$$(x_0, x_1, \ldots, x_n) \equiv (\lambda x_0, \lambda x_1, \ldots, \lambda x_n), \qquad \lambda \in \mathbb{C} \setminus \{0\}. \tag{A.1}$$

This identification lowers the number of dimensions of the space by one. The numbers $(x_0, x_1, \ldots, x_n)$ are called *homogeneous coordinates* on the projective space $\mathbb{P}^n$. It is customary to number the homogeneous coordinates from 0 to n as here. Note that the number of homogeneous coordinates is one more than the number of dimensions.

Equivalent definitions would be to define $\mathbb{P}^n$ as the space of lines through the origin of $\mathbb{C}^{n+1}$, or as the unit sphere in $\mathbb{C}^{n+1}$ with points differing in a common phase $\lambda = e^{i\phi}$ identified.

In the region of $\mathbb{P}^n$ where one selected coordinate x_i is non-zero, one may also define *affine* coordinates

$$(\xi_1, \ldots, \xi_n) = (\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}). \tag{A.2}$$

The solution set of polynomial equations in $\mathbb{C}^n$ or $\mathbb{P}^n$ (in the latter case, such equations must be homogeneous in the homogeneous coordinates x_i) may be a (complex) manifold. More generally, such solution sets are defined as *varieties* in the subject of algebraic geometry. A variety may contain singularities. A projective variety defined by a set of equations f_i is denoted

$$V(f_1, \ldots, f_N) = \{(a_0, \ldots, a_n) \in \mathbb{P}^n \mid f_1(a_0, \ldots, a_n) = \cdots = f_N(a_0, \ldots, a_n) = 0\} \tag{A.3}$$

where $f_1, \ldots, f_N$ are homogeneous polynomials in $\{x_0, \ldots, x_n\}$. One may also define affine varieties by replacing $\mathbb{P}^n$ with $\mathbb{C}^n$.

A.1.2 Algebra and geometry

A *ring* is a set of objects which allow operations of addition and multiplication in a "natural way", i.e., the set is an Abelian group under addition, and multiplication is associative and distributive. Obviously, the sets $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ are rings. $\mathbb{R}$ and $\mathbb{C}$ are also *fields* where the operation of multiplication also gives an Abelian group structure for all nonzero elements.

An important class of rings are the *polynomial rings* $\mathbb{C}[x_1, \ldots, x_n]$. These are the sets of polynomial functions in n variables. They are easily verified to satisfy the conditions for a ring. An *ideal* of a ring is a subring $I \subset R$ such that fg and gf are in I for all $f \in I$ and $g \in R$. For a polynomial ring, an ideal is always finitely generated by a set of functions $(f_1, \ldots, f_m)$, i.e., all $g \in I$ may be written $g = g_i f_i$ for some $g_i \in R$. There is therefore a natural correspondence between an ideal and the variety $V(f_1, \ldots, f_m)$.

An *R-module* M, where R is a ring, is an Abelian group with an additional "natural" operation of multiplication with elements in R, i.e., such that for all $f, g \in R, \alpha, \beta \in M$, the relations $f\alpha \in M$, $f(\alpha + \beta) = f\alpha + f\beta$, $(f + g)\alpha = f\alpha + g\alpha$, and $(fg)\alpha = f(g\alpha)$ hold. A module becomes a vector space if R is a field. Modules, however, are more general and may contain elements α such that $f\alpha = 0$ with $f, \alpha \neq 0$. An ideal of a ring is also a module.

A *finitely generated* module is a module which may be written $M = \sum_i R\alpha_i$ for a set $(\alpha_i) \subset M$. A *free R-module* is a direct sum $M = R^n = \bigoplus_{i=1}^n R$. Any finitely generated module may be written with an exact sequence called a *presentation*

$$R^n \xrightarrow{p} R^m \to M \to 0, \tag{A.4}$$

where the map p may be written as a matrix called the *presentation matrix*. This matrix encodes the relations between the generators of the module. The exact sequence above may be extended to the left by a free module mapping out the kernel of the matrix p. This process may be repeated, possibly indefinitely, or until the leftmost map has no kernel. The result is a *free resolution*,

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0, \tag{A.5}$$

where the F_i are all free modules.

A.2 Bundles and sheaves

Loosely, a *fibre bundle* E is a manifold that locally "looks like" a topological product of two manifolds. One of the factors in this "product" is the *base space* M , and the other is the *fibre* F. There is a projection $\pi : E \to M$ such that $\pi^{-1}(p) \cong F$ for all points $p \in M$. If the fibre has the structure of a vector space, the space E is called a *vector bundle*.

If bundles really always were product spaces, they would not be particularly interesting. The interesting structure to study is in some sense how much a bundle is different from a product space. The case where E is a product of the fibre and the base is called a trivial bundle. Topological information about the "amount of difference" between a bundle and the trivial bundle is contained in *characteristic classes* of the bundle.

The condition that a bundle should be locally similar to a product is formalised in the existence of local diffeomorphisms (called local trivialisations) from the bundle to $U \times F$, where U is an open subset of the base M. An open covering (a set $\{U_i\}$ of open subsets covering the whole space M) with diffeomorphisms $\phi_i : U_i \times F \to \pi^{-1}(U_i)$ such that $\pi\phi_i(p, f) = p$ is called an atlas. The important thing about the atlas is the way any two different maps "connect" in an area where they are both defined. This information is coded in *transition functions* between the maps defined for intersecting subsets.

Where two of the open sets in the covering overlap, $U_i \cap U_j \neq$, the function $t_{ij} = \phi_i^{-1}\phi_j$ is a smooth function $t_{ij} : F \to F$. These functions are the transition functions. As mentioned above, these functions code all the interesting non-trivial data about the bundle. The t_{ij} may be constrained to belong to a group, the *structure group* of the bundle. If the bundle is a vector bundle (of finite rank), so that $F \cong \mathbb{R}^r$, the structure group is a subgroup of GL(r).

If E and M are complex manifolds and the fibre is a complex vector space, E may be a holomorphic vector bundle. The condition for this is that the structure group is $GL(k, \mathbb{C})$ and the transition functions are holomorphic.

A *section* of a bundle E with base space M is a map $s : M \to E$ such that πs is the identity map. For a subset $U \in M$ we can define a *local section* $s : \pi^{-1}(U) \to U$.

A *sheaf* $\mathcal{F}$ may in some sense be associated with a generalisation of a vector bundle. While a vector bundle has a fibre, which is a vector space, over each point in the base space, a sheaf defines a group $\mathcal{F}(U)$ (possibly a vector space) for each open subset U of the base space[1]. The group $\mathcal{F}(U)$ is called the sections of $\mathcal{F}$ over U. In addition, a sheaf defines a map $r_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ for all open subsets $V \subset U$ called the restriction map, which must satisfy

1. For any sequence $U \subset V \subset W$ of open sets,
$$r_{W,U} = r_{V,U} \circ r_{W,V}. \tag{A.6}$$
2. For any pair of open sets $U, V \subset X$ and sections $\sigma \in \mathcal{F}(U)$, $\tau \in \mathcal{F}(V)$ such that $\sigma|_{U \cap V} = \tau|_{U \cap V}$ there exists a section $\rho \in \mathcal{F}(U \cup V)$ with $\rho|_U = \sigma$ and $\rho|_V = \tau$.
3. If $\sigma \in \mathcal{F}(U \cup V)$ and $\sigma|_U = \sigma|_V = 0$, then $\sigma = 0$.

To make the connection to vector bundles, we define the sheaf $\mathcal{O}(E)$ for a holomorphic vector bundle by letting $\mathcal{O}(E)(U)$ be the group of holomorphic sections of E over U. Other important sheaves on a topological space X are $\mathcal{O}_X$, the sheaf of holomorphic functions on (open subsets of) X, and Ω^p_X, the sheaf of holomorphic p-forms on X.

A sheaf is called *locally free* if it is locally isomorphic to $\mathcal{O}^r_X$. It turns out that the category of locally free sheaves is equivalent to the category of vector bundles, in the sense that all locally free sheaves are isomorphic to $\mathcal{O}(E)$ for a vector bundle E.

[1] It is also possible to define a sheaf by its properties at each point of the base space, making the definition more similar to that of a vector space. See e.g. [76].

Bibliography

[1] M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory*, vol. 1. Cambridge University Press, 1987.

[2] J. Polchinski, *String Theory*, vol. II. Cambridge University Press, 1998.

[3] C. V. Johnson, *D-brane primer*, `hep-th/0007170`.

[4] P. H. Ginsparg, *Applied conformal field theory*, in *Fields, Strings and Critical Phenomena* (E. Brézin and J. Zinn-Justin, eds.). Elsevier, 1989. `hep-th/9108028`.

[5] J. Scherk and J. H. Schwarz, *Dual models for nonhadrons*, *Nucl. Phys.* **B81** (1974) 118–144.

[6] M. B. Green and J. H. Schwarz, *Anomaly cancellation in supersymmetric $D = 10$ gauge theory and superstring theory*, *Phys. Lett.* **B149** (1984) 117–122.

[7] M. B. Green and J. H. Schwarz, *Infinity cancellations in SO(32) superstring theory*, *Phys. Lett.* **B151** (1985) 21–25.

[8] F. Gliozzi, J. Scherk and D. I. Olive, *Supersymmetry, supergravity theories and the dual spinor model*, *Nucl. Phys.* **B122** (1977) 253–290.

[9] Y. Nambu, *Quark model and the factorization of the Veneziano amplitude*, in *Symmetries and quark models* (R. Chand, ed.), p. 269. Gordon and Breach, 1970.

[10] T. Goto, *Relativistic quantum mechanics of one-dimensional mechanical continuum and subsidiary condition of dual resonance model*, *Prog. Theor. Phys.* **46** (1971) 1560–1569.

[11] L. Brink, P. Di Vecchia and P. S. Howe, *A locally supersymmetric and reparametrization invariant action for the spinning string*, *Phys. Lett.* **B65** (1976) 471–474.

[12] S. Deser and B. Zumino, *A complete action for the spinning string*, *Phys. Lett.* **B65** (1976) 369–373.

[13] J. Polchinski, *Dirichlet-branes and ramond-ramond charges*, *Phys. Rev. Lett.* **75** (1995) 4724–4727 [hep-th/9510017].

[14] J. L. Cardy, *Conformal invariance and statistical mechanics*, in *Les Houches, Session XLIX, 1988, Fields, strings, critical phenomena* (E. Brézin and J. Zinn-Justin, eds.), pp. 169–245, Elsevier, 1989.

[15] J. Polchinski, *String Theory*, vol. I. Cambridge University Press, 1998.

[16] J. E. Paton and H.-M. Chan, *Generalized veneziano model with isospin*, *Nucl. Phys.* **B10** (1969) 516–520.

[17] M. Bershadsky, C. Vafa and V. Sadov, *D-branes and topological field theories*, *Nucl. Phys.* **B463** (1996) 420–434 [hep-th/9511222].

[18] M. B. Green, J. A. Harvey and G. W. Moore, *I-brane inflow and anomalous couplings on d-branes*, *Class. Quant. Grav.* **14** (1997) 47–52 [hep-th/9605033].

[19] M. Bershadsky, C. Vafa and V. Sadov, *D-strings on D-manifolds*, *Nucl. Phys.* **B463** (1996) 398–414 [hep-th/9510225].

[20] L. Alvarez-Gaume and D. Z. Freedman, *Geometrical structure and ultraviolet finiteness in the supersymmetric sigma model*, *Commun. Math. Phys.* **80** (1981) 443.

[21] H. Ooguri, Y. Oz and Z. Yin, *D-branes on Calabi-Yau spaces and their mirrors*, *Nucl. Phys.* **B477** (1996) 407–430 [hep-th/9606112].

[22] J. Fuchs, C. Schweigert and J. Walcher, *Projections in string theory and boundary states for gepner models*, *Nucl. Phys.* **B588** (2000) 110–148 [hep-th/0003298].

[23] K. Becker, M. Becker and A. Strominger, *Five-branes, membranes and nonperturbative string theory*, *Nucl. Phys.* **B456** (1995) 130–152 [hep-th/9507158].

[24] P. Kaste, *From D-branes on Calabi-Yau manifolds to nonperturbative results in supersymmetric field theory.* PhD thesis, Ruprecht-Karls-Universität, Heidelberg, 2000.

[25] M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory*, vol. 2. Cambridge University Press, 1987.

[26] M. Marino, R. Minasian, G. W. Moore and A. Strominger, *Nonlinear instantons from supersymmetric p-branes*, *JHEP* **01** (2000) 005 [hep-th/9911206].

[27] A. Kapustin and Y. Li, *D-branes in landau-ginzburg models and algebraic geometry*, *JHEP* **12** (2003) 005 [hep-th/0210296].

[28] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, *Large N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183–386 [hep-th/9905111].

[29] J. M. Maldacena, *The large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231–252 [hep-th/9711200].

[30] M. J. Duff and C. N. Pope, *Consistent truncations in kaluza-klein theories*, *Nucl. Phys.* **B255** (1985) 355–364.

[31] J. M. Maldacena and C. Nuñez, *Supergravity description of field theories on curved manifolds and a no go theorem*, *Int. J. Mod. Phys.* **A16** (2001) 822–855 [hep-th/0007018].

[32] M. Bertolini, P. Di Vecchia and R. Marotta, *$N = 2$ four-dimensional gauge theories from fractional branes*, hep-th/0112195.

[33] M. Bertolini, P. Di Vecchia, M. Frau, A. Lerda and R. Marotta, *$N = 2$ gauge theories on systems of fractional D3/D7 branes*, *Nucl. Phys.* **B621** (2002) 157–178 [hep-th/0107057].

[34] M. Bertolini *et. al.*, *Fractional D-branes and their gauge duals*, *JHEP* **02** (2001) 014 [hep-th/0011077].

[35] T. Eguchi and A. J. Hanson, *Selfdual solutions to euclidean gravity*, *Ann. Phys.* **120** (1979) 82.

[36] T. Eguchi, P. B. Gilkey and A. J. Hanson, *Gravitation, gauge theories and differential geometry*, *Phys. Rept.* **66** (1980) 213.

[37] P. S. Aspinwall, *Enhanced gauge symmetries and k3 surfaces*, *Phys. Lett.* **B357** (1995) 329–334 [hep-th/9507012].

[38] M. R. Douglas, *Enhanced gauge symmetry in m(atrix) theory*, *JHEP* **07** (1997) 004 [hep-th/9612126].

[39] C. V. Johnson, A. W. Peet and J. Polchinski, *Gauge theory and the excision of repulson singularities*, *Phys. Rev.* **D61** (2000) 086001 [hep-th/9911161].

[40] R. Friedman, *Algebraic Surfaces and Holomorphic Vector Bundles.* Springer, New York, 1998.

[41] S. K. Donaldson, *Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles*, *Proc. London Math. Soc.* **50** (1985) 1.

[42] K. Uhlenbeck and S. T. Yau, *On the existence of Hermitian-Yang-Mills connections in stable vector bundles*, *Comm. Pure Appl. Math.* **39** (1986) S257.

[43] N. C. Leung, *Einstein type metrics and stability on vector bundles*, *J. Differential Geom.* **45** (1997) 514.

[44] M. R. Douglas, B. Fiol and C. Römelsberger, *Stability and BPS branes*, `hep-th/0002037`.

[45] D.-E. Diaconescu and C. Römelsberger, *D-branes and bundles on elliptic fibrations*, *Nucl. Phys.* **B574** (2000) 245–262 [`hep-th/9910172`].

[46] D. Gepner, *Space-time supersymmetry in compactified string theory and superconformal models*, *Nucl. Phys.* **B296** (1988) 757.

[47] A. Recknagel and V. Schomerus, *D-branes in Gepner models*, *Nucl. Phys.* **B531** (1998) 185–225 [`hep-th/9712186`].

[48] I. Brunner, M. R. Douglas, A. E. Lawrence and C. Römelsberger, *D-branes on the quintic*, *JHEP* **08** (2000) 015 [`hep-th/9906200`].

[49] A. Recknagel, *Permutation branes*, *JHEP* **04** (2003) 041 [`hep-th/0208119`].

[50] E. Cremmer and J. Scherk, *Factorization of the pomeron sector and currents in the dual resonance model*, *Nucl. Phys.* **B50** (1972) 222–252.

[51] M. B. Green and P. Wai, *The insertion of boundaries in world sheets*, *Nucl. Phys.* **B431** (1994) 131–172.

[52] P. Di Vecchia *et. al.*, *Classical p-branes from boundary state*, *Nucl. Phys.* **B507** (1997) 259–276 [`hep-th/9707068`].

[53] P. Di Vecchia and A. Liccardo, *D branes in string theory. i*, *NATO Adv. Study Inst. Ser. C. Math. Phys. Sci.* **556** (2000) 1–59 [`hep-th/9912161`].

[54] J. L. Cardy, *Boundary conditions, fusion rules and the verlinde formula*, *Nucl. Phys.* **B324** (1989) 581.

[55] P. Kaste, W. Lerche, C. A. Lutken and J. Walcher, *D-branes on K3-fibrations*, *Nucl. Phys.* **B582** (2000) 203–215 [`hep-th/9912147`].

[56] A. N. Schellekens and S. Yankielowicz, *Simple currents, modular invariants and fixed points*, *Int. J. Mod. Phys.* **A5** (1990) 2903–2952.

[57] J. I. Latorre and C. A. Lutken, *Constrained* $\mathbb{CP}^n$ *models*, *Phys. Lett.* **B222** (1989) 55.

[58] J. Gates, S. James and T. Hubsch, *Unidexterous locally supersymmetric actions for Calabi-Yau compactifications*, *Phys. Lett.* **B226** (1989) 100.

[59] E. Witten, *Phases of $N = 2$ theories in two dimensions*, *Nucl. Phys.* **B403** (1993) 159–222 [hep-th/9301042].

[60] W. Lerche, C. Vafa and N. P. Warner, *Chiral rings in $n = 2$ superconformal theories*, *Nucl. Phys.* **B324** (1989) 427.

[61] B. R. Greene, *String theory on Calabi-Yau manifolds*, hep-th/9702155.

[62] A. Kapustin and Y. Li, *D-branes in topological minimal models: The landau-ginzburg approach*, *JHEP* **07** (2004) 045 [hep-th/0306001].

[63] I. Brunner, M. Herbst, W. Lerche and B. Scheuner, *Landau-ginzburg realization of open string tft*, hep-th/0305133.

[64] S. K. Ashok, E. Dell'Aquila and D.-E. Diaconescu, *Fractional branes in Landau-Ginzburg orbifolds*, hep-th/0401135.

[65] K. Hori and J. Walcher, *F-term equations near gepner points*, *JHEP* **01** (2005) 008 [hep-th/0404196].

[66] I. Brunner, M. Herbst, W. Lerche and J. Walcher, *Matrix factorizations and mirror symmetry: The cubic curve*, hep-th/0408243.

[67] N. P. Warner, *Supersymmetry in boundary integrable models*, *Nucl. Phys.* **B450** (1995) 663–694 [hep-th/9506064].

[68] D. Eisenbud, *Homological algebra with an application to group representations*, *Trans. Amer. Math. Soc.* **260** (1980) 35–64.

[69] I. Brunner and K. Hori, *Orientifolds and mirror symmetry*, *JHEP* **11** (2004) 005 [hep-th/0303135].

[70] J. Backelin, J. Herzog and H. Sanders, *Matrix factorizations of homogeneous polynomials*, in *Algebra – Some Current Trends* (L. L. Avramov and K. B. Tchakeriam, eds.), no. 1352 in Lecture Notes in Mathematics. Springer, Berlin, 1988.

[71] D. R. Grayson and M. E. Stillman, "Macaulay 2, a software system for research in algebraic geometry." Available at http://www.math.uiuc.edu/Macaulay2/.

[72] M. Nakahara, *Geometry, Topology and Physics.* IOP Publishing, London, 1990.

[73] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry.* Springer, New York, 1995.

[74] P. Griffiths and J. Harris, *Principles of Algebraic Geometry.* Wiley, New York, 1978.

[75] R. Hartshorne, *Algebraic Geometry*, vol. 52 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1977.

[76] K. Kodaira, *Complex Manifolds and Deformation of Complex Structures*, vol. 283 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, New York, 1986.

Part II

Papers

Paper 1

Paper 2

Paper 3